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This Volume of
INSPIRE
is being dedicated to
Aryabhatt: Master Astronomer and Mathematician

Aryabhatt was born in 476 CE (Common Era) in Kusumpur (Bihar). Aryabhatt's intellectual brilliance remapped the boundaries of mathematics and astronomy. In 499 CE, at the age of 23, he wrote a text on astronomy and an unparallel treatise on mathematics called "Aryabhatiyam." Aryabhatt formulated the process of calculating the motion of planets and the time of eclipses. Aryabhatt was the first to proclaim that the earth is round, it rotates on its axis, orbits the sun and is suspended in space – 1000 years before Copernicus published his heliocentric theory. He is also acknowledged for calculating p (Pi) to four decimal places: 3.1416 and the sine table in trigonometry. Centuries later, in 825 CE, the Arab mathematician, Mohammed Ibna Musa credited the value of Pi to the Indians, "This value has been given by the Hindus." And above all, his most spectacular contribution was the concept of zero without which modern computer technology would have been non-existent. Aryabhatt was a colossus in the field of mathematics.

FOREWORD

The present volume of ***INSPIRE*** contains the various research papers of Faculty and Research Scholars of Department of Mathematics, INSTITUTE FOR EXCELLENCE IN HIGHER EDUCATION, BHOPAL (M. P.).

For me it is the realization of a dream which some of us have been nurturing for long and has now taken a concrete shape through the frantic efforts and good wishes of our dedicated band of research workers in our country, in the important area of mathematics.

The editor deserves to be congratulated for this very successful venture. The subject matter has been nicely and systematically presented and is expected to be of use to the workers.

(Dr. M. L. Nath)
Director & Patron
IEHE, Bhopal (M. P.)

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GENERALIZED H-FUNCTION OF TWO VARIABLES, HERMITE POLYNOMIALS AND TIME–DEPENDENT SCHRODINGER EQUATION

By

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ABSTRAT

The aim of this paper is to derive the solution of special one-dimensional time dependent Schrodinger equation involving Hermite polynomials and Generalized H-function of two variables.

1. Introduction

One of the fundamental problems in quantum mechanics is to find solution of Schrodinger equation for different forms of potentials. The Schrodinger equation and more general formulation of quantum mechanics have been set up as a result of the failure of classical physics of predict correctly the result of experiments on microscopic systems; they must be verified by testing their predictions of the properties of systems, where classical mechanics has failed and also where it has succeeded. In fact the whole of atomic physics, solid state physics, chemistry and some other branches of applied sciences obey the principals of quantum mechanics or satisfy differential equations similar to the Schrodinger equations, and the same is almost certainly true of nuclear and particle physics, although the, understanding of very high–energy phenomena, where relativistic effects are important, requires a further generalization of theory.

The Generalized H-Function of two Variables is define by Shrivastava [5], and we will represent here in the following manner

$$H^{m_1, n_1:m_2, n_2;m_3,}_{p_1, q_1:p_2, q_2;p_3, q_3}\left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} (a_j; \alpha_j, A_j)_{1, p1}: (c_j, \gamma_j)_{1, p2}: (e_j, E_j)_{1, p3} \\ (b_j; \beta_j, B_j)_{1, q1}: (d_j, \delta_j)_{1, q2}: (f_j, F_j)_{1, q3} \end{matrix}\right]$$

$$= \frac{-1}{\;} \int_{L_1} \int_{L_2} \phi_1(\xi, \eta)\, \theta_2(\xi)\, \theta_3(\eta) x^{\xi} y^{\eta}\, d\xi\, d\eta, \qquad (1.1)$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n_1} \Gamma(1 - a_j + \alpha_j\xi + A_j\eta) \prod_{j=1}^{m_1} \Gamma(b_j - \beta_j\xi + B_j\eta)}{\prod_{j=n_1+1}^{p_1} \Gamma(a_j - \alpha_j\xi - A_j\eta) \prod_{j=1}^{q_1} \Gamma(1 - b_j + \beta_j\xi + B_j\eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_2} \Gamma(d_j - \delta_j\xi) \prod_{j=1}^{n_2} \Gamma(1 - c_j + \gamma_j\xi)}{\prod_{j=m_2+1}^{q_2} \Gamma(1 - d_j + \delta_j\xi) \prod_{j=n_2+1}^{p_2} \Gamma(c_j - \gamma_j\xi)},$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(f_j - F_j\eta) \prod_{j=1}^{n_3} \Gamma(1 - e_j + E_j\eta)}{\prod_{j=m_3+1}^{q_3} \Gamma(1 - f_j + F_j\eta) \prod_{j=n_3+1}^{p_3} \Gamma(e_j - E_j\eta)},$$

2. Integral

Making an appeal of [1, p.1, (1.2)], we can obtain the following integrals:

$$\int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_{2\nu}(x)\, H[z x^{2h}, y]\, dx$$

$$= 2^{2\nu} \; H^{m_1, n_1:\; m_2, n_2+2;\; m_3,}_{p_1, q_1: p_2+2, q_2+1; p_3, q_3} \left[\begin{matrix} z \\ y \end{matrix} \middle| \begin{matrix} \ldots\ldots : (1/2-\rho, h), (-\rho, h), \ldots : \ldots\ldots \\ \ldots\ldots : \ldots\ldots, (\nu-\rho, h) : \ldots\ldots \end{matrix} \right] \quad (2.1)$$

$$\int_{-\infty}^{\infty} x^{2\rho+1} e^{-x^2} H_{2\nu+1}(x)\, H[z x^{2h}, y]\, dx$$

$$= 2^{2\rho+1} \; H^{m_1, n_1:\; m_2, n_2+2;\; m_3,}_{p_1, q_1: p_2+2, q_2+1; p_3, q_3} \left[\begin{matrix} z \\ y \end{matrix} \middle| \begin{matrix} \ldots\ldots : (-1/2-\rho, h), (-\rho, h), \ldots : \ldots\ldots \\ \ldots\ldots : \ldots\ldots, (\nu-\rho, h) : \ldots\ldots \end{matrix} \right]. \quad (2.2)$$

valid under the conditions, which are given in section 1.

3. The Special Schrodinger Equation

We consider the problem of a particle in the potential V(x) given by

$$V(x) = [h^2 / (2m)]\, x^2 \qquad (3.1)$$

The time dependent Schrodinger equation [4,p.16, (2.10)] for this system can therefore ,be written in the form.

$$\frac{\partial u}{\partial t} = \frac{-h}{2im}\frac{\partial^2 u}{\partial x^2} + \frac{h}{2im}(x^2 u) \qquad (3.2)$$

Setting K = – h /(2im) into (3.2) , we have

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} - Kx^2 u, \qquad (3.3)$$

provided u(x,t) tend to 0 for large values of t and $|x| \to \infty$. we also assume that

$$u(x, 0) \equiv u(x). \qquad (3.4)$$

The solution of (3.2) is given by Bajpai [2, p. 168, (3.1), (3.2)], as under:

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-k(2n+1)t - x^2/2} He_n(\sqrt{2}\, x),$$

where $He_n(x)$ are Chebyshev Hermite polynomials [3, p.166 -168]

$$u(x,t) = \sum_{n=0}^{\infty} B_n e^{-k(2n+1)t - x^2/2} H_n(x), \qquad (3.5)$$

where $H_n(x)$ are Hermite polynomials.

$$\text{also } A_n = 1/(n!\sqrt{\pi}) \int_{-\infty}^{\infty} u(x) e^{-x^2/2} He_n(\sqrt{2}x)\, dx \qquad (3.6)$$

$$B_n = 1/(2^n n!\sqrt{2}) \int_{-\infty}^{\infty} u(x) e^{-x^2/2} H_n(x)\, dx \qquad (3.7)$$

4. Solution in Terms of H-function of Two Variables

The solution of (3.5) leads to the following solutions:

$$u_1(x, t) = \sum_{n=0}^{\infty} B_{2n} e^{-k(4n+1)t - x^2/2} H_{2n}(x), \qquad (4.1)$$

where

$$B_{2n} = 1/[2^{2n}(2n)!\sqrt{\pi}) \int_{-\infty}^{\infty} u_1(x)\, e^{-x^2/2} H_{2n}(x)\, dx \qquad (4.2)$$

$$u_2(x, t) = \sum_{n=0}^{\infty} B_{2n+1} e^{-k(4n+3)t - x^2/2} H_{2n+1}(x)\, dx \qquad (4.3)$$

where

$B_{2n+1} = 1/[2^{2n+1}(2n+1)!\sqrt{\pi}] \quad \int_{-\infty}^{\infty} u_2(x)\; e^{-x^2/2}\, H_{2n+1}(x)\, dx \quad (4.4)$

If we substitute

$$u_1(x) = x^{2\rho}\, e^{-x^2/2}\; H[z\, x^{2h}, y] \qquad (4.5)$$

and

$$u_2(x) = x^{2\rho+1}\, e^{-x^2/2}\, H[z\, x^{2h}, y] \qquad (4.6)$$

in (4.2) and (4.4) respectively and use the integrals (2.1) and (2.2), then the solutions corresponding to (4.1) and (4.3) are given by.

$$u_1(x, t) = 1/(\sqrt{\pi}) \sum_{n=1}^{\rho} [1/(2n)!]\; e^{-k(4n+1)t - x^2/2}$$

$$\times\; H^{m_1, n_1:\; m_2, n_2+2;\; m_3,}_{p_1, q_1: p_2+2, q_2+1; p_3, q_3} \left[\begin{matrix} z \\ y \end{matrix} \middle| \begin{matrix} \text{..........}: (1/2-\rho, h), (-\rho, h), \text{......}: \text{..........} \\ \text{............}: \text{...................}, (\nu-\rho, h): \text{.........} \end{matrix} \right] H_{2n}(x) \qquad (4.7)$$

valid under the conditions of (2.1).

$$u_2(x, t) = 1/(\sqrt{\pi}) \sum_{n=1}^{\rho} [1/(2n+1)!]\; e^{-k(4n+3)t - x^2/2}$$

$$\times H^{m_1, n_1:\; m_2, n_2+2;\; m_3,}_{p_1, q_1: p_2+2, q_2+1; p_3, q_3} \left[\begin{matrix} z \\ y \end{matrix} \middle| \begin{matrix} \text{..........}: (-1/2-\rho, h), (-\rho, h), \text{......}: \text{.........} \\ \text{............}: \text{...................}, (\nu-\rho, h): \text{.........} \end{matrix} \right] H_{2n+1}(x) \qquad (4.8)$$

valid under the conditions of (2.2).

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I-FUNCTION AND ELECTROSTATIC POTENTIAL IN SPHERICAL REGIONS

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Abstract

In this paper first we evaluate an integral involving I-function of one variable and then we make its application to solve a boundary value problem related to electrostatic potential in spherical regions.

1. Introduction:

The I-function of one variable is defined by Saxena [1, p.366-375] and we will represent here in the following manner:

$$I^{m,n}_{p_i, q_i : r}\left[x \left| \begin{matrix} [(a_j, \alpha_j)_{1,n}], [(a_{ji}, \alpha_{ji})_{n+1}, p_i] \\ [(b_j, \beta_j)_{1,m}], [(b_{ji}, \beta_{ji})_{m+1}, q_i] \end{matrix} \right.\right] = (1/2\pi i) \int_L \theta(s)\, x^s\, ds \qquad (1)$$

where $i = \sqrt{(-1)}$,

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r} \left(\prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} s) \right)},$$

integral is convergent, when ($B>0$, $A \leq 0$), where

$$B = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{pi} \alpha_{ji} + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{qi} \beta_{ji}, \qquad (2)$$

$$A = \sum_{j=1}^{pi} \alpha_{ji} - \sum_{j=1}^{qi} \beta_{ji}, \qquad (3)$$

$$|\arg x| < \tfrac{1}{2} B\pi,\ \forall\ i \in (1, 2, \ldots, r). \qquad (4)$$

2. Result Required:

The following result are required in our present investigation:

From Erdelyi [2]:

$$\int_{-1}^{1} (1 + x)^{(\lambda - 1)} P_\nu(x)dx = \frac{2^\lambda [\Gamma(\lambda)]^2}{\Gamma(\lambda + \nu + 1)\, \Gamma(\lambda - \nu)}, \tag{5}$$

Re $(\lambda) > 0$;

3. Integral:

The integral to be established is:

$$\int_{-1}^{1} (1 + x)^{\alpha - 1} P_l (x)\, I^{m, n}_{p_i, q_i: r} [z (1 + x)^\lambda]dx$$

$$= 2^\alpha I^{m, n + 2}_{p_i + 2, q_i + 2: r} \left[z2^\lambda \middle| \begin{matrix} (1 - \alpha, \lambda), (1 - \alpha, \lambda), \ldots. \\ \ldots.., (- \alpha - l, \lambda), (1 + l - \alpha, \lambda) \end{matrix} \right], \tag{6}$$

provided that Re$(\alpha) > 0$, $\lambda \geq 0$, $(\alpha - l)$ is not a negative integer and $|\arg z| < \frac{1}{2}\pi B$, where B is given in (2).

To prove (6), replace the I-function by its equivalent contour integral as given in (1), change the order of integration which is valid under the given condition, evaluate the inner integral with the help of (5) and finally interpret it with (1), to get (6).

4. Problem on Electrostatic Potential in Spherical Regions:

Let us now consider the harmonic function V representing the electrostatic potential in the domain $r < c$ such that V assumes a prescribed value $F(\theta)$ on the spherical surface $r = c$, where r, ϕ and θ are the special polar co-ordinates and V is independent or ϕ. Thus V satisfies Laplace equation

$$r \frac{\partial^2 (rV)}{\partial r^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0 \tag{7}$$

in the domain $r < c$, $0 < \theta < \pi$ and under the condition

$$\lim_{r \to 0} V = F(\theta), \ (0 < \theta < \pi, r = c), \tag{8}$$

where V and its partial derivatives of first and second orders are assumed to be continuous through the interior of the sphere:

$0 \le r < c, 0 \le \theta < \pi.$ (9)

Physically the function V may represent steady temperature in a solid sphere $r \le c$, where surface temperature depends only on θ; i.e. the surface temperature is uniform over each circle $\theta = \theta_0$, r = c. Here V also denotes electrostatic potential in the space r < c free of charge, for V = F(θ) on the boundary r = c.

If we take cos θ = x ($0 \le \theta < \pi$), the equation (7) reduces to

$$r \frac{\partial^2 (rV)}{\partial r^2} + \frac{\partial}{\partial x} (1 - x^2) \frac{\partial V}{\partial x} = 0, \quad r < c, -1 < x < 1. \tag{10}$$

If further we take F(θ) = f(cos θ) = f(x), then V(r, x) satisfies the transformed equation (10) with the boundary condition

$$\lim_{r \to c} V(r, x) = f(x), (r < c, -1 < x < 1), \tag{11}$$

where V is continuous every where interior to the sphere and bounded when $0 \le r \le r_0 < c$.

Here in (11), we may take

$$f(x) = (1 + x)^{\alpha - 1} I^{m, n}_{p_i, q_i : r} [z (1 + x)^{\lambda}], \tag{12}$$

where Re (α) > 0, $\lambda \ge 0$ and |arg z| < ½ πB, where B is given in (2) and f(x), f ´(x) are assumed to be sectionally continuous over the interval (– 1, 1).

$$\lim_{r \to \infty} W(r, x) = 0, \tag{13}$$

where W is the harmonic function in the unbounded domain r > c, exterior to the spherical surface and rV is bounded for large value of r and for all x ($-1 \le x \le 1$).

Solution of the Problem:

Case I: Solution for V(r, x) when r < c (Interior to the sphere)

In this case solution of the problem is given by

$$V(r, x) = \sum_{S=0}^{\infty} R_s (r/c)^s P_s(x) \ (r < c). \tag{14}$$

Making an appeal of (11), the above relation (14) reduces to

$$f(x) = \sum_{S=0}^{\infty} R_s P_s(x). \tag{15}$$

Now making an appeal to orthogonal property of Legendre polynomials Erdelyi [2, p.277 (13)] and integral (6), from (12) and (15) we obtain

$$R_l = (2l+1)\, 2^{\alpha-1}\, I^{m,\, n+2}_{p_i+2,\, q_i+2:\, r}\left[z\, 2^{\lambda} \middle| \begin{matrix} (1-\alpha, \lambda), (1-\alpha, \lambda), \ldots \\ \ldots, (-\alpha - l, \lambda), (1 + l - \alpha, \lambda) \end{matrix}\right], \tag{16}$$

provided that all the conditions of (6) are satisfied.

Making an appeal to (16), from (14) we obtain the required relation

$$V(r, x) = 2^{\alpha-1} \sum_{S=0}^{\infty} (2s+1)\, I^{m,\, n+2}_{p_i+2,\, q_i+2:\, r}\left[z\, 2^{\lambda} \middle| \begin{matrix} (1-\alpha, \lambda), (1-\alpha, \lambda), \ldots \\ \ldots, (-\alpha - s, \lambda), (1 + s - \alpha, \lambda) \end{matrix}\right]$$

$$\cdot\, (r/c)^{s} P_s(x), \tag{17}$$

where all the conditions of (6) are satisfied.

Case II: Solution for W(r, x) when $r \geq c$ (Exterior to the spherical surface)

We consider the following solution of the problem similar to Churchill [3, p.219, eqn. (12)]:

$$W(r, x) = \sum_{S=0}^{\infty} R_s\, (r/c)^{s+1} P_s(x), \; (r \geq c). \tag{18}$$

In this case solution of the problem is given by

$$W(r, x) = 2^{\alpha-1} \sum_{S=0}^{\infty} (2s+1)\, I^{m,\, n+2}_{p_i+2,\, q_i+2:\, r}\left[z\, 2^{\lambda} \middle| \begin{matrix} (1-\alpha, \lambda), (1-\alpha, \lambda), \ldots \\ \ldots, (-\alpha - s, \lambda), (1 + s - \alpha, \lambda) \end{matrix}\right]$$

$$\cdot\, (r/c)^{s+1} P_s(x), \tag{19}$$

where all the conditions of (6) are satisfied.

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SEMICOMPATIBILITY AND FIXED POINT THEOREMS ININTUITIONISTIC FUZZY METRIC SPACE USING IMPLICIT RELATION

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Abstract

The concept of semi compatibility has been introduced in intuitionistic fuzzy metric space and it has been applied to prove results on existence of unique common fixed point of four self mappings satisfying an implicit relation. Recently, poppa has employed a similar but not the same implicit relation to obtain a fixed point theorem for d-complete topological spaces all the results of this paper are new and will generalized the Result of Bijendra singh and Shishir jain [26].

1. Introduction

The concept of fuzzy sets was introduced by *zadeh* [13] following the concept of fuzzy sets, fuzzy metric spaces have been introduced by kramosil and michlek [9] and George and veeramani [6] modified the notion of fuzzy metric space with the help of continuous t-norms.

As a generalization of fuzzy sets , Atanassov [14] introduced and studied the concept of intuitionistic fuzzy sets park[11] using the idea of intuitionistic fuzzy sets defined the notion of intuionistic fuzzy metric spaces with the help of continuous t-norm and continuous t co-norm as a generalization of fuzzy metric space due to George & veeramani [6] had showed that every metric induces an intuitionistic fuzzy metric every fuzzy metric space is an intuitionistic fuzzy metric space and found a necessary and sufficient condition for an intuitionistic fuzzy metric space to be complete choudhary[15] introduced mutually contractive sequence of self maps and proved a fixed point theorem kramosil & michlek [9] introduced the notion of Cauchy sequences in an intuitionistic fuzzy metric space and proved the well known fixed point theorem of Banach [4].

Turkoglu et al [12] gave the generalization of jungek's common fixed point theorem [19] to intuitionistic fuzzy metric spaces, they first formulate the definition of weakly commuting and R- weakly commuting mapping in intuitionistic fuzzy metric spaces and proved proved the intuitionistic fuzzy version of pant's theorem[20].

So we define a semi compatible pair of self maps in intuitionistic fuzzy metric space. Saliga [29] and Sharma et.al [30] proved some interesting fixed point results using implicit real functions and semi compatibility in d-complete topological spaces. Recently Poppa in [28] used the family F_4 of implicit real function to find the fixed points of two pairs of semi compatible maps in d-complete topological space. here, F_4 denote the family of all real continuous functions $F: (\mathbb{R}^4) \to \mathbb{R}$ satisfying the following properties.

(F_1) There exists $h \geq 1$ such that for every $u \geq 0, v \geq 0$ with $F(u,v,u,v) \geq 0$ or $F(u,v,v,u)$ we have $u \geq hv$

$(F_2) F(u,u,0,0) < 0$, For all $u > 0$

Here we will introduce an implicit relation in IFM-space and prove some results using semi-compatibility .Our Result will generalizes the results [26] and [27].

2. Preliminaries

Definition 2.1 [7] A binary operation $*\,[0,1] \times [0,1] \to [0,1]$ is a continuous t norm if it satisfies the following condition

2.1.1 $*$ is commutative and associative

2.1.2 $*$ is continuous

Address of corresponding author

2.1.3 $a * 1 = a$ for all $a \in [0,1]$

2.1.4 $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$

Definition 2.2 [7] A binary operation $\lozenge: [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\lozenge$ is satisfying the following condition

2.2.1 $\lozenge$ is commutative and associate

2.2.2 $\lozenge$ is continuous

2.2.3 $a \lozenge 0 = a$ for all $a \in [0,1]$

2.2.4 $a \lozenge b \leq c \lozenge d$ Whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$

Definition 2.3 [4] the 3-tuple (X, M, $*$) is called a fuzzy metric space (FM-space) if X is an arbitrary set $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,\infty]$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$.

2.3.1 $M(x,y,0) > 0$

2.3.2 $M(x,y,t) = 1, \forall\, t > 0 \; iff \; x = y$

2.3.3 $M(x,y,t) = M(y,x,t)$,
2.3.4 $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$
2.3.5 $M(x,y,\ .): [0,\infty] \to [0,1]$ is continuous.

Remark 2.4 since $*$ is continuous, it follows from (2.3.4) that the limit of a sequence in FM-space is uniquely determined

Definition 2.5 [16] A five –tuple (X, M, N, $*$, ◊) is said to be an intuitionistic fuzzy metric space if X is an arbitrary set $*$ is a continuous t – norm, ◊ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0,\infty)$ satisfying the following conditions for all $x,y,z \in X, s,t > 0$

2.4.1 $M(x,y,t) + N(x,y,t) \leq 1$
2.4.2 $M(x,y,t) > 0$
2.4.3 $M(x,y,t) = M(y,x,t)$
2.4.4 $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$
2.4.5 $M(x,y,.): (0,\infty) \to (0,1)$ is continuous
2.4.6 $N(x,y,t) > 0$
2.4.7 $N(x,y,t) = N(y,x,t)$
2.4.8 $N(x,y,t) \lozenge N(y,z,s) \geq N(x,z,t+s)$
2.4.9 $N(x,y,.): (0,\infty) \to (0,1]$ is continuous

Then (M, N) is called an intuitionistic fuzzy metric On X , the function $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non- nearness between x and y with respect to t respectively

Remark 2.6 Every fuzzy metric space (X, M, $*$) is an intuitionistic fuzzy metric space form$(X, M, 1-M, *, \lozenge)$ such that t-norm $*$ and t-conorm ◊ are associated ie $x \lozenge y = 1 - ((1-x) * (1-y))$ for any $x, y \in [0,1]$ but the converse is not true

Definition 2.7 two maps F and G from intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ into itself are said to be R-weakly commuting if there exist a positive real number R such that for each $x \in X$

$$M(FGx, GFx, Rt) \geq M(Fx, Gx, t)$$

and

$$N(FGx, GFx, Rt) \leq M(Fx, Gx, t) \text{ for all } t > 0$$

Definition 2.8 two self mappings F and G of an intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ are said to be compatible if $lim_{n\to\infty} M(FGx_n, GFx_n, t) = 1, lim_{n\to\infty} N(FGx_n, GFx_n, t) = 0$

Definition 2.9 two self maps F and G of intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ are said to be weak compatible if they commute at their coincidence points that is $Fx = Gx$ implies $FGx = GFx$

Definition 2.10 A pair (F,G) of self maps of an intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ are said to be semi-compatible if $lim_{n\to\infty} FGx_n = Gx$ whenever $\{x_n\}$ is a sequence in X such that

$$lim_{n\to\infty} Fx_n = lim_{n\to\infty} Gx_n = x$$

It follows that (F,G) is semi-compatible and $Fx = Gx$ then $FGx = GFx$

Remark 2.11 from the above definitions it is clear that in an intuitionistic fuzzy metric space a pair (F,G) of self maps is R-weakly commuting implies that the pair is compatible, which implies that the pair is weak-compatible. But the converse is not true

Example 2.12 let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space where $X = [0,2]$ t norm is defined $a * b = \min\{a, b\}$ and t co-norm is defined by $a \lozenge b = \max\{a, b\}$ for all $a, b \in [0,2]$ and $M(x, y, t) = [\exp(\frac{|x-y|}{t})^{-1}]$ and

$N(x, y, t) = \left[\exp\left(\frac{|x-y|}{t}\right) - 1\right] [\exp\left(\frac{|x-y|}{t}\right)^{-1}]$ for all $x, y \in X, t > 0$

Define self maps F and G on X as follow

$$Fx = \begin{cases} x^2 + 1 & 0 \le x < 1 \\ 3 & 1 \le x \le 3 \end{cases}$$

$$Gx = \begin{cases} 1-x & 0 \le x < 1 \\ 3 & 1 \le x \le 3 \end{cases}$$

Take $\{x_n\} = \frac{1}{2^n}$, then $lim_{n\to\infty} x_n = 0$ and $lim_{n\to\infty} Fx_n = lim_{n\to\infty} Gx_n = 1$

Now $M(FGx_n, GFx_n, t) = [\exp\left(\frac{FGx_n - GFx_n}{t}\right)^{-1}]$

$= [\exp\left(\left|2 + \frac{1}{2^{2n}} - \frac{2}{2^n} - 3\right|\right)]^{-1} \neq 1$ and

$N(FGx_n, GFx_n, t) = [\exp(\frac{|FGx_n - GFx_n|}{t}) - 1][\exp\left(\frac{FGx_n - GFx_n}{t}\right)^{-1}]$

$\to [\exp(\frac{-1}{t}) - 1][\exp\left(\frac{-1}{t}\right))^{-1}] \neq 0$ as $n \to \infty$

Hence F and G are not compatible .The set of coincident points of F and G is {1,3} Now for any

$x \in \{1,3\}, Fx = Gx = 3$ and $= F(3) = 3 = GFx = G(3)$. Thus F and G are weak-compatible but not compatible.

Lemma 2.13[24] in intuitionistic fuzzy metric space X, $M(x, y, .)$ is non decreasing and $N(x, y, .)$ is non increasing for all $x, y \in X$

Lemma 2.14[24] let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space if there exist $k \in (0,1)$ such that $M(x, y, kt) \ge M(x, y, t)$ and $N(x, y, kt) \le N(x, y, t)$ for all $x, y \in X$ Then $x = y$.

Proof: $\because M(x, y, kt) \ge M(x, y, t)$

and

$N(x,y,kt) \leq N(x,y,t)$

Then we have

$M(x,y,t) \geq M(x,y,\frac{t}{k})$

and

$N(x,y,t) \leq N(x,y,\frac{t}{k})$

By repeated application of above inequality as we have

$M(x,y,t) \geq M\left(x,y,\frac{t}{k}\right) \geq M\left(x,y,\frac{t}{k^2}\right) \geq ----M\left(x,y,\frac{t}{k^n}\right) \geq ------$

and

$N(x,y,t) \leq N\left(x,y,\frac{t}{k}\right) \leq N\left(x,y,\frac{t}{k^2}\right) \leq -------N\left(x,y,\frac{t}{k^n}\right) \leq ----$

For $n \in N$ which tends to 1 and 0 as $n \to \infty$ respectively thus

$M(x,y,t) = 1$

and

$N(x,y,t) = 0$

For all $t > 0$ and we get $x = y$

Lemma 2.15[16]. Let $\{y_n\}$ be a sequence in IFM space $(X,M,N,*,\lozenge)$ with the condition $lim_{t\to\infty}M(x,y,t) = 1$ and $lim_{t\to\infty}N(x,y,t) = 0$. If there exist a number $k \in (0,1)$ such that $M(y_{2n+2,}y_{n+1},kt) \geq M(y_{2n+1,}y_n,t)$ and $(y_{2n+2,}y_{n+1},kt) \leq N(y_{2n+1,}y_n,t)$, for all $t > 0$, then $\{y_n\}$ is a Cauchy sequence in X

Lemma 2.16. Let A and B be two self-maps on a complete IFM-space $(X,M,N,*,\lozenge)$ such that for some k $\in (0,1)$, for all $x,y \in X$, for all $t > 0$

$M(Ax,Bx,t) \geq \min\{M(x,y,\text{t}),M(Ax,x,t)\}$

and

$N(Ax,Bx,t) \leq \max\{N(x,y,\text{t}),N(Ax,x,t)\}$

Then A and B have a unique common fixed point in X.

Proof Let $p \in X$. taking $x_0 = p$, define sequence $\{x_n\}$ in X by $Ax_{2n} = x_{2n+1}$ and $x_{2n+1} = x_{2n+2}$. By taking x $= x_{2n}, y = x_{2n+1}$ *and* $x = x_{2n}, y = x_{2n-1}$, respectively, in the contractive condition , we obtain that

$M(x_{n+1},x_n,kt) \geq M(x_n,x_{n-1},\text{t}),$

and

$N(x_{n+1},x_n,kt) \leq N(x_n,x_{n-1},\text{t})$ *for all* $t > 0$, for all n

Therefore by lemma 2.7, $\{x_n\}$ is a Cauchy sequence in X, which is complete. Hence $\{x_n\}$ converges to some u in X. Taking $x = x_{2n}$ and $y = u$ and letting $n \to \infty$ in the contractive condition, we get $= u$. Similarly, by putting $x = u$ and $y = x_{2n+1,}$ we get $Au = u$. Therefore, u is the common fixed point of the maps A and B. The uniqueness of the common fixed point follows from the contractive condition.

Proposition2.17 [**13**]. Let A and S be self maps on a IFM- space $(X,M,N,*,\lozenge)$. If S is continuous, then (A,S) is semi compatible if and only if (A,S) is compatible.

2.18: A class of implicit relation $\Phi_4 and \Psi_4$ be the set of all real continuous function, $\Phi_4, \Psi_4: (\mathbb{R}^4) \to \mathbb{R}$ decreasing and increasing in second, third and fourth argument respectively and satisfying the following conditions

(i) For every $u, v \in (0,1)$with $\Phi(u^3, v^3, v^3, u^3) \geq 0$ or $\Phi(u^3, v^3, v^3, v^3) \geq 0$ or $\Phi(u^3, u^3, u^3, v^3) \geq 0$ we have $u^3 \geq v^3$ and for Ψ_4

(ii) For every $u, v \in (0,1)$ with $\Psi(u^3, v^3, v^3, u^3) \geq 0$ or $\Psi(u^3, v^3, v^3, v^3) \geq 0$ or $\Psi(u^3, u^3, u^3, v^3) \geq 0$ we have $u^3 \geq v^3$

Example: Define $\Phi\ ({t_1}^3, {t_3}^3, {t_3}^3, {t_3}^3,) = {t_1}^3 - max\{{t_2}^3, {t_3}^3, {t_3}^3,\}$
and

$$\Psi({t_1}^3, {t_3}^3, {t_3}^3, {t_3}^3,) = min\ \{{t_2}^3, {t_3}^3, {t_3}^3\} - {t_1}^3$$

3. Main Result

Theorem 3.1. Let A, B, S, and T be self -mappings of a complete IFM-space $(X, M, N, *, \Diamond)$ satisfying that

(3.1) $A(X) \subseteq T(X), B(X) \subseteq S(X)$

(3.2) the pair (A, S) is semi compatible and (B, T) is weak compatible

(3.3) one of A or S is continuous

For some $\varphi \in \Phi_4$ and $\Psi \in \Psi_4$, there exist $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$,

(3.4) $\varphi\begin{pmatrix} M^3(Ax, By, kt), M^3(Sx, Ty, t), \\ M^3(Ax, Sx, t), M^3(By, Ty, kt) \end{pmatrix} \geq 0$

(3.5) $\varphi\begin{pmatrix} M^3(Ax, By, kt), M^3(Sx, Ty, t), \\ M^3(Ax, Sx, kt), M^3(By, Ty, t) \end{pmatrix} \geq 0$

and

(3.6) $\Psi\begin{pmatrix} N^3(Ax, By, kt), N^3(Sx, Ty, t), \\ N^3(Ax, Sx, t), N^3(By, Ty, kt) \end{pmatrix} \geq 0$

(3.7) $\Psi\begin{pmatrix} N^3(Ax, By, kt), N^3(Sx, Ty, t), \\ N^3(Ax, Sx, kt), N^3(By, Ty, t) \end{pmatrix} \geq 0$

Then A, B, S, and T have unique common fixed point in X.

Proof . Let $x_0 \in X$ be any arbitrary point by (3.1) we can choose a sequence $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 0,1,2,3 \ldots \ldots \ldots ..$, now using (3.4) and (3.6) with $x = x_{2n}, y = x_{2n+1}$, we get

$$\varphi\begin{pmatrix} M^3(Ax_{2n}, Bx_{2n+1}, kt), M^3(Sx_{2n}, Tx_{2n+1}, t), \\ M^3(Ax_{2n}, Sx_{2n}, t), M^3(Bx_{2n+1}, Tx_{2n+1}, kt) \end{pmatrix} \geq 0$$

$$\varphi\begin{pmatrix} M^3\left(y_{2n+1}, y_{2n+2}, kt\right), M^3\left(y_{2n}, y_{2n+1}, t\right), \\ M^3\left(y_{2n+1}, y_{2n}, t\right), M^3\left(y_{2n+2}, y_{2n+1}, kt\right) \end{pmatrix} \geq 0$$

and

$$\Psi\begin{pmatrix}N^3(Ax_{2n},Bx_{2n+1},kt),N^3(Sx_{2n},Tx_{2n+1},t),\\N^3(Ax_{2n},Sx_{2n},t),N^3(Bx_{2n+1},Tx_{2n+1},kt)\end{pmatrix}\geq 0$$

$$\Psi\begin{pmatrix}N^3\left(y_{2n+1},y_{2n+2},kt\right),N^3\left(y_{2n},y_{2n+1},t\right),\\N^3\left(y_{2n+1},y_{2n},t\right),N^3\left(y_{2n+2},y_{2n+1},kt\right)\end{pmatrix}\geq 0$$

Using (i) and (ii) of 2.18 we get

$$M^3(y_{2n+2},y_{2n+1},kt)\geq M^3(y_{2n+1},y_{2n},t)$$

This implies

$$N(y_{2n+2},y_{2n+1},kt)\geq N(y_{2n+1},y_{2n},t)$$

and

$$N^3(y_{2n+2},y_{2n+1},kt)\leq N^3(y_{2n+1},y_{2n},t)$$

This implies

$$N(y_{2n+2},y_{2n+1},kt)\leq N(y_{2n+1},y_{2n},t)$$

Similarly by putting $x = x_{2n+2}, y = x_{2n+1}$ in (3.5) and (3.7) we have

$$\varphi\begin{pmatrix}M^3(Ax_{2n+2},Bx_{2n+1},kt),M^3(Sx_{2n+2},Tx_{2n+1},t),\\M^3(Ax_{2n+2},Sx_{2n+2},kt),M^3(Bx_{2n+1},Tx_{2n+1},t)\end{pmatrix}\geq 0$$

$$\varphi\begin{pmatrix}M^3\left(y_{2n+3},y_{2n+2},kt\right),M^3\left(y_{2n+2},y_{2n+1},t\right)\\M^3\left(y_{2n+3},y_{2n+2},kt\right),M^3\left(y_{2n+2},y_{2n+1},t\right)\end{pmatrix}\geq 0$$

and

$$\Psi\begin{pmatrix}N^3(Ax_{2n+2},Bx_{2n+1},kt),N^3(Sx_{2n+2},Tx_{2n+1},t),\\N^3(Ax_{2n+2},Sx_{2n+2},kt),N^3(Bx_{2n+1},Tx_{2n+1},t)\end{pmatrix}\geq 0$$

$$\Psi\begin{pmatrix}N^3\left(y_{2n+3},y_{2n+2},kt\right),N^3\left(y_{2n+2},y_{2n+1},t\right)\\N^3\left(y_{2n+3},y_{2n+2},kt\right),N^3\left(y_{2n+2},y_{2n+1},t\right)\end{pmatrix}\geq 0$$

Using (i) and (iii) of 2.10 we get

$$M^3(y_{2n+3},y_{2n+2},kt)\geq M^3(y_{2n+1},y_{2n+2},t)$$

This implies that

$$M(y_{2n+3},y_{2n+2},kt)\geq M(y_{2n+1},y_{2n+2},t)$$

and

$$N^3(y_{2n+3},y_{2n+2},kt)\leq N^3(y_{2n+1},y_{2n+2},t)$$

This implies that

$$N(y_{2n+3},y_{2n+2},kt)\leq N(y_{2n+1},y_{2n+2},t)$$

Thus for any n and t , we have

$$M(y_n,y_{n+1},kt)\geq M(y_{n-1},y_n,t)$$

and

$$N(y_n,y_{n+1},kt)\leq N(y_{n-1},y_n,t)$$

Hence by lemma (2.7), $\{y_n\}$ is a Cauchy sequence in X, which is complete. Therefore $\{y_n\}$ converges to $u \in X$. Its sub –sequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}, \{Tx_{2n+1}\}$, also converges to u, that is $\{Ax_{2n}\} \to u, \{Bx_{2n+1}\} \to u, \{Sx_{2n}\} \to u, \{Tx_{2n+1}\} \to u$.

Case (I) (S is continuous), in this case we have

$$SAx_{2n} \to Su, S^2x_{2n} \to Su$$

The semi-compatibility of the pair (A, S) gives

$$lim_{n\to\infty} ASx_{2n} = Su$$

Step I by putting $x = Sx_{2n}, y = x_{2n+1}$ in (3.4) and (3.6) we obtain that

$$\varphi\begin{pmatrix} M^3(ASx_{2n}, Bx_{2n+1}, kt), M^3(SSx_{2n}, Tx_{2n+1}, t), \\ M^3(ASx_{2n}, SSx_{2n}, t), M^3(Bx_{2n+1}, Tx_{2n+1}, kt) \end{pmatrix} \geq 0$$

and

$$\Psi\begin{pmatrix} N^3(ASx_{2n}, Bx_{2n+1}, kt), N^3(SSx_{2n}, Tx_{2n+1}, t), \\ N^3(ASx_{2n}, SSx_{2n}, t), N^3(Bx_{2n+1}, Tx_{2n+1}, kt) \end{pmatrix} \geq 0$$

Letting $n \to \infty$, and by the continuity of t-norm $*$, and t-conorm $\Diamond$, we have

$$\varphi\begin{pmatrix} M^3(Su, u, kt), M^3(Su, u, t), \\ M^3(Su, Su, t), M^3(Su, u, kt) \end{pmatrix} \geq 0$$

$$\varphi\begin{pmatrix} M^3(Su, u, kt), M^3(Su, u, t), \\ 1,1 \end{pmatrix} \geq 0$$

Since φ is decreasing in third and fourth argument therefore

$$\varphi\begin{pmatrix} M^3(Su, u, kt), M^3(Su, u, t), \\ M^3(Su, u, t), M^3(Su, u, t) \end{pmatrix} \geq 0$$

and

$$\Psi\begin{pmatrix} N^3(Su, u, kt), N^3(Su, u, t), \\ N^3(Su, Su, t), N^3(Su, u, kt) \end{pmatrix} \geq 0$$

$$\Psi\begin{pmatrix} N^3(Su, u, kt), N^3(Su, u, t), \\ 0,0 \end{pmatrix} \geq 0$$

Since Ψ is increasing in third and fourth argument therefore

$$\Psi\begin{pmatrix} N^3(Su, u, kt), N^3(Su, u, t), \\ N^3(Su, u, t), N^3(Su, u, t) \end{pmatrix} \geq 0$$

Using (i) and (ii) of (2.18) we get

$M^3(Su, u, kt) \geq M^3(Su, u, t)$ *and* $N^3(Su, u, kt) \leq N^3(Su, u, t)$

This implies $M(Su, u, kt) \geq M(Su, u, t)$and $N(Su, u, kt) \leq N(Su, u, t)$

For all $t > 0$, therefore by lemma2.14 we have

$$Su = u$$

Step II. By putting $x = u, y = x_{2n+1}$ in (3.4) and (3.6) we obtain that

$$\varphi\begin{pmatrix} M^3(Au, Bx_{2n+1}, kt), M^3(Su, Tx_{2n+1}, t), \\ M^3(Au, Su, t), M^3(Bx_{2n+1}, Tx_{2n+1}, kt) \end{pmatrix} \geq 0$$

Taking limit as $n \to \infty$, we get

$$\varphi\begin{pmatrix} M^3(Au,u,kt), M^3(u,u,t), \\ M^3(Au,u,t), M^3(u,u,kt) \end{pmatrix} \geq 0$$

$$\varphi\,[\,M^3(Au,u,kt), M^3(Au,u,t)1,1,)] \geq 0$$

Since φ is decreasing in second and fourth argument therefore

$$\varphi\,[\,M^3(Au,u,kt), M^3(Au,u,t), M^3(Au,u,t), M^3(Au,u,t),)] \geq 0$$

and

$$\Psi\begin{pmatrix} N^3(Au,Bx_{2n+1},kt), N^3(Su,Tx_{2n+1},t), \\ N^3(Au,Su,t), N^3(Bx_{2n+1},Tx_{2n+1},kt) \end{pmatrix} \geq 0$$

Taking limit as $n \to \infty$, we get

$$\Psi\begin{pmatrix} N^3(Au,u,kt), N^3(u,u,t), \\ N^3(Au,u,t), N^3(u,u,kt) \end{pmatrix} \geq 0$$

$$\Psi\,[\,N^3(Au,u,Kt), 0, N^3(Au,u,t), 0,)] \geq 0$$

Since Ψ is increasing in second and fourth argument therefore

$$\Psi\,[\,N^3(Au,u,Kt), N^3(Au,u,t), N^3(Au,u,t), N^3(Au,u,t),) \geq 0]$$

Using (i) and (ii) of 2.17 we have

$M^3(Au,u,Kt) \geq M^3(Au,u,t)$ *and* $N^3(Au,u,Kt) \leq N^3(Au,u,t)$for all $t > 0$
this implies $M(Au,u,Kt) \geq M(Au,u,t)$*and* $N(Au,u,Kt) \leq N(Au,u,t)$
therefore by lemma2.14 we have $u = Au$, i.e $Au = Su = u$

Step 3 As $(X) \subset T(X)$. There exist $w \in X$ such that $Au = Su = u = Tw$. by putting $x = x_{2n,} y = w$ in (3.4) and (3.6), we obtain that

$$\varphi\begin{pmatrix} M^3(Ax_{2n},Bw,kt), M^3(Sx_{2n},Tw,t), \\ M^3(Ax_{2n},Sx_{2n},t), M^3(Bw,Tw,kt) \end{pmatrix} \geq 0$$

Taking limit as $n \to \infty$, we get

$$\varphi\,[\,M^3(u,Bw,kt), 1,1, M^3(u,Bw,kt),] \geq 0$$

Since φ is decreasing in second and third argument therefore

$$\varphi\,[\,M^3(u,Bw,kt), M^3(u,Bw,t), M^3(u,Bw,t), M^3(u,Bw,kt),] \geq 0$$

and

$$\Psi\begin{pmatrix} N^3(Ax_{2n},Bw,kt), N^3(Sx_{2n},Tw,t), \\ N^3(Ax_{2n},Sx_{2n},t), N^3(Bw,Tw,kt) \end{pmatrix} \geq 0$$

Taking limit as $n \to \infty$, we get

$$\Psi\,[\,N^3(u,Bw,kt), 0,0, N^3(u,Bw,kt),] \geq 0$$

Since Ψ is increasing in second and third argument therefore

$$\Psi\,[\,N^3(u,Bw,kt), N^3(u,Bw,t), N^3(u,Bw,t), N^3(u,Bw,kt),] \geq 0$$

Using (i) and (ii) of 2.18 we have

$M^3(u,Bu,kt) \geq M^3(u,Bw,t)$ *and* $N^3(u,Bu,kt) \leq N^3(u,Bw,t)$ for all $t > 0$
this implies that $M(u,Bu,kt) \geq M(u,Bw,t)$ *and* $N(u,Bu,kt) \leq N(u,Bu,t)$
Therefore by lemma2.14 $u = Bw$ .Therefore $Bw = Tw = u$
Since (B , T) is weak-compatible we get that $TBw = BTw$ that is $Bu = Tu$

Step 4. By putting $x = u, y = v$ in condition (3.4) and (3.6) we obtain that

$$\varphi\begin{pmatrix} M^3(Au, Bv, kt), M^3(Su, Tv, t), \\ M^3(Au, Su, t), M^3(Bv, Tv, kt) \end{pmatrix} \geq 0$$

That is

$$\varphi\,[\,M^3(Au, Bv, kt), M^3(Au, Bv, t), 1,1] \geq 0$$

Since φ is decreasing in third and fourth argument therefore

$$\varphi\,[\,M^3(Au, Bv, kt), M^3(Au, Bv, t), M^3(Au, Bv, t), 1] \geq 0$$

and

$$\Psi\begin{pmatrix} N^3(Au, Bv, kt), N^3(Su, Tv, t), \\ N^3(Au, Sv, t), N^3(Bv, Tv, kt) \end{pmatrix} \geq 0$$

$$\Psi\,[\,N^3(Au, Bu, kt), N^3(Au, Bu, t), 0,0,] \geq 0$$

Since Ψ is increasing in third and fourth argument therefore

$$\Psi\,[\,N^3(Au, Bu, kt), N^3(Au, Bu, t), N^3(Au, Bu, t), N^3(Au, Bu, t),] \geq 0$$

Using (ii) and (iv) of 2.17 we have

$M^3(Au, Bu, kt) \geq M^3(Au, Bu, t)$ *and* $N^3(Au, Bu, kt) \leq N^3(Au, Bu, t)$ for all > 0 this implies that $M(Au, Bu, kt) \geq M(Au, Bu, t)$ *and* $N(Au, Bu, kt) \leq N(Au, Bu, t)$

Therefore by lemma2.14 we have $Bu = Au$.

Therefore $u = Au = Su = Bu = Tu$ that is u is a common fixed point of A, B, S, and T.

Case II (A is continuous) in this case we have $ASx_{2n} \rightarrow Au$

The semi-compatibility of the pair (A, S) gives

$$ASx_{2n} \rightarrow Su$$

By uniqueness of limit in IFM-space, we obtain that

$$Au = Su$$

Step 5 By putting $x = u, y = x_{2n+1}$ in condition (3.4) and (3.6) we obtain that

$$\varphi\begin{pmatrix} M^3(Au, Bx_{2n+1}, kt), M^3(Su, Tx_{2n+1}, t), \\ M^3(Au, Su, t), M^3(Bx_{2n+1}, Tx_{2n+1}, kt) \end{pmatrix} \geq 0$$

Taking limit as $n \rightarrow \infty$, and we get

$$\varphi\,[\,M^3(Au, u, kt), 1, M^3(Au, u, t), 1] \geq 0$$

Since φ is decreasing in secondand fourth argument therefore

$$\varphi\,[\,M^3(Au, u, kt), M^3(Au, u, t), M^3(Au, u, t), M^3(Au, u, t)] \geq 0$$

and

$$\Psi\begin{pmatrix} N^3(Au, Bx_{2n+1}, kt), N^3(Su, Tx_{2n+1}, t), \\ N^3(Au, Su, t), N^3(Bx_{2n+1}, Tx_{2n+1}, kt) \end{pmatrix} \geq 0$$

Taking limit as $n \rightarrow \infty$, and we get

$$\Psi\,[\,N^3(Au, u, kt), 0, N^3(Au, u, t), 0,] \geq 0$$

Since Ψ is increasing in second and fourth argument therefore

$$\Psi\,[\,N^3(Au, u, kt), N^3(Au, u, t), N^3(Au, u, t), N^3(Au, u, t),] \geq 0$$

Using (i) and (iii) of 2.17 we have

$M^3(Au,u,kt) \geq M^3(Au,u,t)$ *and* $N^3(Au,u,t) \leq N^3(Au,u,t)$ For all $t > 0$, this implies that

$$M(Au,u,kt) \geq M(Au,u,t) \text{ and } N(Au,u,t) \leq N(Au,u,t)$$

Therefore from lemma2.14 we have $u = Au$, and the rest of the proof follows from step3, onwards of the previous case.

Uniqueness let z, be another common fixed point of A,B, S, and T then $z = Az = Sz = Bz = Tz$, putting $x = u$ *and* $y = z$ in (3.4) and (3.6), we get

$$\varphi\begin{pmatrix} M^3(Au,Bz,kt), M^3(Su,Tz,t), \\ M^3(Au,Su,t), M^3(Bz,Tz,kt) \end{pmatrix} \geq 0$$

That is

$$\varphi\,[\,M^3(u,z,kt), M^3(u,z,t), 1,1,] \geq 0$$

Since φ is decreasing in third and fourth argument therefore

$$\varphi\,[\,M^3(u,z,kt), M^3(u,z,t), M^3(u,z,t), M^3(u,z,t),] \geq 0$$

and

$$\Psi\begin{pmatrix} N^3(Au,Bz,kt), N^3(Su,Tz,t), \\ N^3(Au,Su,t), N^3(Bz,Tz,kt) \end{pmatrix} \geq 0$$

That is

$$\Psi\,[\,N^3(u,z,kt), N^3(u,z,t), 0,0,] \geq 0$$

Since Ψ is increasing in third and fourth argument therefore

$$\Psi\,[\,N^3(u,z,kt), N^3(u,z,t), N^3(u,z,t), N^3(u,z,t),] \geq 0$$

Using (i) and (ii) of 2.18 we have

$M^3(u,z,kt) \geq M^3(u,z,t)$ *and* $N^3(u,z,t) \leq N^3(u,z,t)$ For all $t > 0$ therefore by lemma 2.14 we have $u = z$.

Therefore u is the unique common fixed point of the self maps A, B, S and T.

Corollary3.2. Let A, B, S and T be self mappings of a complete IFM-space $(X, M, N, *, \Diamond)$ satisfying conditions (3.1), (3.4), (3.5), (3.6) and (3.7) and that the pairs (A,S) and (B,T) are semi-compatible.

One of A, B, S, or T is continuous

Then A, B, S, and T have a unique common fixed point in X

Proof. As semi-compatibility implies weak-compatibility the proof follows from theorem 3.1

On taking $A = B$ in theorem 3.1 we have the following corollary

Corollary3.3. Let A, B, S, and T is self-mappings of a complete IFM-space $(X, M, N, *, \Diamond)$ satisfying that

$$A(X) \subset T(X) \cap S(X)$$

The pair (A, S) is semi-compatible and (A, T) is weak compatible. One of A or S is continuous.

For some $\Phi \in \Phi_4$ and $\Psi \in \Psi_4$ there exists $k \in (0,1)$ such that for all, $y \in X$, and $t > 0$

$$\varphi\begin{pmatrix}M^3(Ax,By,kt),M^3(Sx,Ty,t),\\M^3(Ax,Sx,t),M^3(By,Ty,kt)\end{pmatrix}\geq 0$$
$$\varphi\begin{pmatrix}M^3(Ax,By,kt),M^3(Sx,Ty,t),\\M^3(Ax,Sx,kt),M^3(By,Ty,t)\end{pmatrix}\geq 0$$

and

$$\Psi\begin{pmatrix}N^3(Ax,By,kt),N^3(Sx,Ty,t),\\N^3(Ax,Sx,t),N^3(By,Ty,kt)\end{pmatrix}\geq 0$$
$$\Psi\begin{pmatrix}N^3(Ax,By,kt),N^3(Sx,Ty,t),\\N^3(Ax,Sx,kt),N^3(By,Ty,t)\end{pmatrix}\geq 0$$

Then A, S and T, have a unique common fixed point in X

Proof Taking $S = I$ and $T = I$ in theorem 3.1 the conditions (3.1), (3.2), and (3.3) are satisfied trivially and we get the following corollary.

Theorem3.4. Let A, B, S and T be self mappings of a complete IFM-space $(X, M, N,*,\Diamond)$ satisfying conditions (3.1), (3.4), (3.5), (3.6) and (3.7) and that the pair (A, S) is compatible and (B, T) is weak compatible. Then A, B, S and T have a unique common fixed point in X

Proof. In view of proposition 2.13 and theorem 3.1 it suffices to prove the Result, when A is continuous as in the proof of theorem 3.1 the sequence $\{y_n\} \to u \in X$, As A is continuous, we have

$$ASx_{2n} \to Au, AAx_{2n} \to Au$$

The compatibility of (A, S) gives

$$lim_{n\to\infty}ASx_{2n} = Su = lim_{n\to\infty}SAx_{2n}$$

Step I By putting $x = Ax_{2n}$, and $y = x_{2n+1}$ in condition (3.4) we get that

$$\varphi\begin{pmatrix}M^3(AAx_{2n},Bx_{2n+1},kt),M^3(SAx_{2n},Tx_{2n+1},t),\\M^3(AAx_{2n},SAx_{2n},t),M^3(Bx_{2n+1},Tx_{2n+1},kt)\end{pmatrix}\geq 0$$

and

$$\Psi\begin{pmatrix}N^3(AAx_{2n},Bx_{2n+1},kt),N^3(SAx_{2n},Tx_{2n+1},t),\\N^3(AAx_{2n},SAx_{2n},t),N^3(Bx_{2n+1},Tx_{2n+1},kt)\end{pmatrix}\geq 0$$

Taking limit as $n \to \infty$, and we get

$$\varphi\begin{pmatrix}M^3(Au,u,kt),M^3(Au,u,t),\\M^3(Au,Au,t),M^3(u,u,kt)\end{pmatrix}\geq 0$$

and

$$\Psi\begin{pmatrix}N^3(Au,u,kt),N^3(Au,u,t),\\N^3(Au,Au,t),N^3(u,u,kt)\end{pmatrix}\geq 0$$

That is

$$\varphi\,[\,M^3(Au,u,kt),M^3(Au,u,t),1,1,]\geq 0$$

Since φ is decreasing in third and fourth argument therefore

$$\varphi\,[\,M^3(Au,u,kt),M^3(Au,u,t),M^3(Au,u,t),M^3(Au,u,t),]\geq 0$$

and

$\Psi\,[\,N^3(Au,u,kt), N^3(Au,u,t), 0{,}0,\,] \geq 0$

Since Ψ is increasing in third and fourth argument therefore

$\Psi\,[\,N^3(Au,u,kt), N^3(Au,u,t), N^3(Au,u,t), N^3(Au,u,t),\,] \geq 0$

Using (ii) and (iv), we have

$M^3(Au,u,t) \geq M^3(Au,u,t)$ *and* $N^3(Au,u,t) \leq N^3(Au,u,t)$ For all $t > 0$

This implies that $M(Au,u,t) \geq M(Au,u,t)$ *and* $N(Au,u,t) \leq N(Au,u,t)$

Therefore from lemma2.14 $u = Au$

Step II As $A(X) \subset T(X)$ there exist $w \in X$ such that $Au = Su = u = Tw$ by putting $x = x_{2n}, y = w$ in (3.4) and (3.5) we obtain that

$$\varphi\begin{pmatrix} M^3(Ax_{2n}, Bw, kt), M^3(Sx_{2n}, Tw, t), \\ M^3(Ax_{2n}, Sx_{2n}, t), M^3(Bw, Tw, kt) \end{pmatrix} \geq 0$$

and

$$\Psi\begin{pmatrix} N^3(Ax_{2n}, Bw, kt), N^3(Sx_{2n}, Tw, t), \\ N^3(Ax_{2n}, Sx_{2n}, t), N^3(Bw, Tw, kt) \end{pmatrix} \geq 0$$

Taking limit as $n \to \infty$, and we get

$\varphi\,[\,M^3(u,Bw,kt), 1{,}1, M^3(Bw,u,kt),\,] \geq 0$

Since φ is decreasing in second and third argument therefore

$\varphi\,[\,M^3(u,Bw,kt), M^3(Bw,u,kt), M^3(Bw,u,kt), M^3(Bw,u,kt),\,] \geq 0$

and

$\Psi\,[\,N^3(u,Bw,kt), 0{,}0, N^3(Bw,u,kt),\,] \geq 0$

Since Ψ is increasing in second and third argument therefore

$\Psi\,[\,N^3(u,Bw,kt), N^3(Bw,u,t), N^3(Bw,u,t), N^3(Bw,u,kt),\,] \geq 0$

Using (i) and (iii) of 2.17 we have

$M^3(u,Bw,kt) \geq M^3(u,Bw,t)$ *and* $N^3(u,Bw,kt) \leq N^3(Bw,u,t)$ for all $t > 0$

this implies that $M(u,Bw,kt) \geq M(u,Bw,t)$ *and* $N(u,Bw,kt) \leq N(Bw,u,t)$

Therefore from lemma2.1$u = Bw$ therefore $Bw = Tw = u$ as (B.T) is weak compatible we have $TBw = BTw$ that is $Bu = Tu$

Step III again as $u = Bw$ and $(X) \subset S(X)$, there exist $v \in X$ such that $Bw = Sv = u$ by putting $x = v, y = w$ in (3.4) and (3.6) we have

$$\varphi\begin{pmatrix} M^3(Av, Bw, kt), M^3(Sv, Tw, t), \\ M^3(Av, Sv, t), M^3(Bw, Tw, kt) \end{pmatrix} \geq 0$$

and

$$\Psi\begin{pmatrix} N^3(Av, Bw, kt), N^3(Sv, Tw, t), \\ N^3(Av, Sv, t), N^3(Bw, Tw, kt) \end{pmatrix} \geq 0$$

That is

$\varphi\,[\,M^3(Av,Sv,kt), 1, M^3(Av,Sv,t), 1] \geq 0$

Since φ is decreasing in second and fourth argument therefore

$\varphi\,[\,M^3(Av,Sv,kt), M^3(Av,Sv,t), M^3(Av,Sv,t), M^3(Av,Sv,t)] \geq 0$

and

$\Psi\,[N^3(Av,Sv,kt),0,N^3(Av,Sv,t),0]\geq 0$

Since Ψ is increasing in second and fourth argument therefore
$\Psi\,[N^3(Av,Sv,kt),N^3(Av,Sv,t),N^3(Av,Sv,t),N^3(Av,Sv,t)]\geq 0$
Using (i) and (iii) of 2.17 we have
$M^3(Av,Sv,kt)\geq M^3(Av,Sv,kt)\ and\ N^3(Av,Sv,kt)\leq N^3(Av,Sv,t)$ For all $t>0$ this implies that $M(Av,Sv,kt)\geq M(Av,Sv,kt)\ and\ N(Av,Sv,kt)\leq N(Av,Sv,t)$ Therefore from lemma2.1 $Av=Sv$, as (A,S) is compatible we have $ASv=SAv$ or $Au=Su=u$ also $Au=Bu$ follows from step 4 in the proof of theorem 3.1 and it follows that u is a common fixed point of the four maps A, B, S, and T, the uniqueness follows as in the proof of theorem 3.1.

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ANGULAR DISPLACEMENT IN A SHAFT INVOLVING H–FUNCTION

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Abstract

The object of this section is to employ Fox's H-function in obtaining a solution of the partial differential equation

$$\frac{\partial^2\theta}{\partial t^2} = b^2 \frac{\partial^2\theta}{\partial x^2}$$

of angular displacement in a shaft. The result yields a number of particular cases on specializing the parameters and may prove to be useful in several interesting situations.

1. Introduction:

The H-function of one variable [1, p.10] is defined as:

$$H_{p,\,q}^{m,\,n}\left[x \,\middle|\, \begin{matrix}(a_j, \alpha_j)_{1,\,p} \\ (b_j, \beta_j)_{1,\,q}\end{matrix}\right] = (1/2\pi i) \int_L \theta(s)\, x^s\, ds \tag{1}$$

where $i = \sqrt{(-1)}$,

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s)}$$

where

$$\sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j \equiv M > 0, \tag{2}$$

and $|\arg x| < ½ M\pi$.

2. Integral:

The integral to be evaluated here is

$$\int_0^{\mu} \cos(\frac{\pi\epsilon x}{\mu})\,(\sin\frac{\pi x}{2\mu})^{2\epsilon-\sigma-1}(\cos\frac{\pi x}{2\mu})^{\sigma-1} H_{p,q}^{m,n}[z(\tan\frac{\pi x}{2\mu})^{2h}\,|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}]dx$$

$$= \frac{\mu 2^{2\epsilon-\sigma}}{\sqrt{\pi}\Gamma(2\epsilon)} H_{p+2,q+1}^{m+1,n+1}[z/4^h|_{(\sigma,2h),(b_j,\beta_j)_{1,q}}^{\left(1-\epsilon+\frac{\sigma}{2},h\right),(a_j,\alpha_j)_{1,p},(\frac{1}{2}-\epsilon+\frac{\sigma}{2},h)}] \quad (3)$$

provided that $2\epsilon > Re\left(\sigma - 2h\frac{b_j}{\beta_j}\right) > 0$, $|\arg z| < ½ M\pi$, where M is given in (2).

Proof: To establish (3), express the H-function as given in (1) into the Mellin-Barnes type contour integrals, then interchange the order of summation and integration, which is justified due to the absolute convergence of the integral involved in the process and then evaluate the inner integral with the help of following result

$$\int_0^{\mu} \cos(\frac{\pi\epsilon x}{\mu})\,(\sin\frac{\pi x}{2\mu})^{2\epsilon-\sigma-1}(\cos\frac{\pi x}{2\mu})^{\sigma-1}dx$$

$$= \frac{\mu 2^{2\epsilon-\sigma}\Gamma(\frac{2\epsilon-\sigma}{2})\Gamma(\sigma)}{\sqrt{\pi}\Gamma(\frac{1-2\epsilon+\sigma}{2})\Gamma(2\epsilon)},\ 2\epsilon > Re(\sigma) > 0 \quad (4)$$

which can be obtained with a little simplification from the result [2, p.375, 3.634 {eqns(2)}].

Finally, interpreting by virtue of (1), we arrive at the desired result.

3. Formulation of the Problem:

As an example of the application of the H-function in applied mathematics we shall consider the problem of determining the angular displacement or twist $\theta(x,t)$ in a shaft of circular section with its axis along the x-axis. If the ends x = 0 and x = μ of the shaft are free, the displacement $\theta(x,t)$ due to initial twist must satisfy the boundary value problem

$$\frac{\partial^2\theta}{\partial t^2} = b^2\frac{\partial^2\theta}{\partial x^2} \quad (5)$$

$$\frac{\partial}{\partial x}\theta(0,t) = 0, \frac{\partial}{\partial x}\theta(\mu,t) = 0, \frac{\partial}{\partial t}\theta(x,0) \quad (6)$$

$$\theta(x,0) = f(x) \quad (7)$$

where b is a constant.

The solution of the problem can be written as Churchill [3, p.125(4)]:

$$\theta(x,t) = \frac{1}{2}a_0 + \sum_{l=1}^{\infty} a_l \cos\frac{\pi l x}{\mu} \cos\frac{l\pi b t}{\mu} \qquad (8)$$

where $a_l(l = 0,1,2,\ldots.)$ are the coefficient in the Fourier cosine series for f(x) in the interval (0,μ).

4. Solution of the Problem:

Here we shall consider

$$f(x) = (\sin\frac{\pi x}{2\mu})^{2\epsilon-\sigma-1}(\cos\frac{\pi x}{2\mu})^{\sigma-1}H_{p,q}^{m,n}[z(\tan\frac{\pi x}{2\mu})^{2h} |_{(b_j,\beta_j)_{1,q}}^{(a_j,\propto_j)_{1,p}}] \qquad (9)$$

If t = 0, then by virtue of (9), we have

$$(\sin\frac{\pi x}{2\mu})^{2\epsilon-\sigma-1}(\cos\frac{\pi x}{2\mu})^{\sigma-1}H_{p,q}^{m,n}[z(\tan\frac{\pi x}{2\mu})^{2h} |_{(b_j,\beta_j)_{1,q}}^{(a_j,\propto_j)_{1,p}}]$$

$$= \frac{1}{2}a_0 + \sum_{l=1}^{\infty} a_l \cos\frac{\pi l x}{\mu}. \qquad (10)$$

Now multiplying both side of (10) by $\cos(\frac{\pi\epsilon x}{\mu})$ and integrating with respect to x from 0 to μ, we get

$$\int_0^{\mu}\cos(\frac{\pi\epsilon x}{\mu})\,(\sin\frac{\pi x}{2\mu})^{2\epsilon-\sigma-1}(\cos\frac{\pi x}{2\mu})^{\sigma-1}H_{p,q}^{m,n}[z(\tan\frac{\pi x}{2\mu})^{2h} |_{(b_j,\beta_j)_{1,q}}^{(a_j,\propto_j)_{1,p}}]dx$$

$$= \frac{1}{2}a_0\int_0^{\mu}\cos(\frac{\pi\epsilon x}{\mu})\,dx + \sum_{l=1}^{\infty} a_l \int_0^{\mu}\cos(\frac{\pi\epsilon x}{\mu})\cos\frac{\pi l x}{\mu}\,dx. \qquad (11)$$

Now using (3) and the orthogonality property of the cosine function, we have

$$a_l = \frac{2^{2l-\sigma+1}}{\sqrt{\pi}\Gamma(2l)} H_{p+2,q+1}^{m+1,n+1}[z/4^h|_{(\sigma,2h),(b_j,\beta_j)_{1,q}}^{\left(1-l+\frac{\sigma}{2},h\right),(a_j,\propto_j)_{1,p},(\frac{1}{2}-l+\frac{\sigma}{2},h)}] \qquad (12)$$

With the help of (8) and (12), we obtained the following solution of the problem:

$$\theta(x,t) = \frac{1}{2^{\sigma}\sqrt{\pi}}\sum_{l=1}^{\infty}\frac{2^{2l+1}}{\Gamma(2l)}\cos\frac{\pi l x}{\mu}\cos\frac{l\pi b t}{\mu}$$

$$\times H_{p+2,q+1}^{m+1,n+1}[z/4^h|_{(\sigma,2h),(b_j,\beta_j)_{1,q}}^{\left(1-\epsilon+\frac{\sigma}{2},h\right),(a_j,\propto_j)_{1,p},(\frac{1}{2}-\epsilon+\frac{\sigma}{2},h)}] \qquad (13)$$

provided that $Re(\sigma) > 0$, |arg z| < ½ Mπ, where M is given in (2).

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PROBLEM RELATED TO COOLING OF A SPHERE INVOLVING I-FUNCTION

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Abstract

In this paper first we evaluate an integral involving I-function of one variable and then we make its application to solve a boundary value problem related to problem related to cooling of a sphere.

1. Introduction:

The I-function of one variable is defined by Saxena [1, p.366-375] and we will represent here in the following manner:

$$I^{m,n}_{p_i, q_i: r}\left[x \,\middle|\, \begin{matrix}[(a_j, \alpha_j)_{1,n}], [(a_{ji}, \alpha_{ji})_{n+1}, p_i] \\ [(b_j, \beta_j)_{1,m}], [(b_{ji}, \beta_{ji})_{m+1}, q_i]\end{matrix}\right] = (1/2\pi i)\int_L \theta(s)\, x^s\, ds \quad (1)$$

where $i = \sqrt{(-1)}$,

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r} \left(\prod^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} s) \right)},$$

integral is convergent, when ($B>0$, $A \le 0$), where

$$B = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p_i} \alpha_{ji} + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q_i} \beta_{ji}, \quad (2)$$

$$A = \sum_{j=1}^{p_i} \alpha_{ji} - \sum_{j=1}^{q_i} \beta_{ji}, \quad (3)$$

$$|\arg x| < \tfrac{1}{2} B\pi, \ \forall\ i \in (1, 2, \ldots, r). \quad (4)$$

We have observed that the separation of variables method can be applied to several types of problems on bounded spatial domains. The problem must be linear and have homogeneous boundary conditions. In this section we present the solution to a classical problem in three dimensions, the cooling of a sphere. The assumed symmetries in the problem will permit us to reduce the dimension of the problem to one spatial dimension and time.

Given a sphere whose initial temperature depends only on the distance from the center (e.g., a constant initial temperature) and whose boundary is kept at a constant temperature, predict the temperature at any point inside the sphere at a later time. This is the problem, for example, of determining the temperature of the center of a potato that has been put in a hot oven. The reader might also conjecture that this problem is important for medical examiners who want to determine the time of an individual's death. Early researchers, notably Kelvin, used this problem to determine the age of the earth based on assumptions about its initial temperature and its temperature today.

This cooling problem may remind the reader of Newton's law of cooling, which is encountered in ordinary differential equations texts. Recall that this law states that the rate at which a body cools is proportional to the difference of its temperature and the temperature of the environment. Quantitatively, if T = T(t) is the temperature of a body and T_e is the temperature of its environment, then T '(t) = K(T_e − T), where K is the constant of proportionality. But the reader should note that this law applies only in the case that the body has a uniform, homogeneous temperature. In the PDE problem we are considering, the temperature may vary rapidly throughout the body.

2. Result Required:

In the present investigation we require the following results:

From Gradshteyn [2]:

$$\int_0^{\pi}(\sin x)^{s-1}\sin nx\,dx = \frac{\pi \sin\frac{n\pi}{2}\Gamma(s)}{2^{s-1}\Gamma(\frac{s \pm n+1}{2})}, \qquad (5)$$

where, Re (s) > 0.

3. Integral:

The integral, which we need as follows:

$$\int_0^{\pi} (\sin x)^{\omega-1} \sin nx\, I^{m,\, l}_{p_i,\, q_i:\, r}[z\,(\sin x)^{-\lambda}]\, dx$$

$$= 2^{1-\omega}\,\pi \sin\frac{n\pi}{2}\; I^{\,m+1,\,l}_{\,p_i+2,\,q_i+1:\,r}\left[z\,2^{\lambda}\,\middle|\,\begin{matrix}\ldots\ldots,(1/2+\omega/2 \pm n/2\,,\,\lambda/2)\\(\omega,\lambda),\ldots\ldots\end{matrix}\right], \qquad (6)$$

provided that $\lambda > 0$, $\omega > 0$ and $|\arg z| < ½ B\pi$, where B is given in (2).

To prove (6), replace the I-function by its equivalent contour integral as given in (1), change the order of integration which is valid under the given condition, evaluate the inner integral with the help of (5) and finally interpret it with (1), to get (6).

Formulate the Problem:

For simplicity, let us consider a sphere of radius $\rho = \pi$ whose initial temperature is T_0 = constant. We will assume that the boundary is held at zero degrees for all time t > 0. If u is the temperature, then in general u will depend on three spatial coordinates and time. But a little reflection shows that the temperature will depend only on the distance from the center of the sphere and on time. Evidently, the temperature u must satisfy the heat equation

$$u_t = k\Delta u,$$

where k is the diffusivity and Δ is the Laplacian. It should be clear that a spherical coordinate system ρ, ϕ, θ is more appropriate than a rectangular system, and $u = u(\rho, t)$. Because u does not depend on the angles ϕ and θ, the Laplacian takes on a particularly simple form:

$$\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial}{\partial\rho}$$

Therefore, we may formulate the model as

$$u_t = k(u_{\rho\rho} + \frac{2}{\rho}u\rho),\ 0 \leq \rho < \pi,\ t > 0. \qquad (7)$$

$$u(\pi, t) = 0,\ t > 0, \qquad (8)$$

$$u(\rho, 0) = T_0,\ 0 \leq \rho < \pi. \qquad (9)$$

Observe that there is an implied boundary condition at $\rho = 0$, namely that the temperature should remain bounded.

To solve problem (7) – (9) we assume $u(\rho, t) = y(\rho)g(t)$. Substituting into the PDE and separating variables gives, following solution, which is given by Logan [3, p. 144 – 146]:

$$u(\rho, t) = \sum_{n=1}^{\infty} c_n\, e^{-n^2kt}\,\frac{\sin n\rho}{\rho} \qquad (10)$$

where

$$c_n = \frac{2}{\pi}\int_0^{\pi} T_0\,\rho \sin n\rho\, d\rho \qquad (11)$$

Solution of the Problem:

The solution of the problem to be obtained is

$$u(\rho, t) = \sum_{n=1}^{\infty} e^{-n^2kt} \frac{\sin n\rho}{\rho}$$

$$. \ 2^{2-\omega} \sin\frac{n\pi}{2} \ I^{m+1, l}_{p_i+2, q_i+1: r} \left[z \, 2^{\lambda} \left| \begin{matrix} \ldots, (1/2+\omega/2 \pm n/2, \lambda/2) \\ (\omega, \lambda), \ldots \end{matrix} \right. \right], \qquad (12)$$

where all conditions of convergence are same as in (6).

Proof:

Choose

$$f(\rho) = u(\rho, 0) = T_0 = \rho^{-1} (\sin\rho)^{\omega-1} \ I^{m, l}_{p_i, q_i: r} [z (\sin\rho)^{-\lambda}]. \qquad (13)$$

Now combining (13) and (11) and making the use of the integral (6), we derive

$$c_n = 2^{2-\omega} \sin\frac{n\pi}{2} \ I^{m, l+1}_{p_i+1, q_i+2: r} \left[z \, 2^{\lambda} \left| \begin{matrix} \ldots\ldots, (1/2+\omega/2 \pm n/2, \lambda/2) \\ (\omega, \lambda), \ldots \end{matrix} \right. \right], \qquad (14)$$

Putting the value of C_n from (14) in (10), we get the required result (12).

Special Cases:

We notice that the I-function involved in our result (12) is quite general character; indeed by suitable specializing the parameters of this function, we can easily obtain solution of the problem in terms of a large number of elementary functions of one or more variables (or product of several such functions).

On specializing the parameters, I-function may be reduced to H-function, G-function, Lauricella's functions Legendre functions, Bessel functions, hypergeometric functions, Appell's functions, Kampe de Feriet's functions and several other higher transcendental functions. Therefore the result (12) is of general nature and may reduced to be in different forms, which will be useful in the literature on applied Mathematics and other branches.

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APPLICATION OF G-FUNCTION IN THE STUDY OF TIME-DOMAIN SYNTHESIS PROBLEM

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Abstract

The object of this paper is to evaluate an integral involving Bessel polynomial and G-function and employ it to obtain a particular solution of the general solution derived in this chapter of the classical problem known as the, "time-domain synthesis problem", occurring in electrical network theory. On specializing the parameters, G-function may be reduced to almost all elementary functions and special function appearing in applied mathematics [1, pp. 215-222]. Therefore the special solution given in the paper is of a general character and hence may encompass several cases of interest.

1. Introduction:

Most of the practical problems, where elementary functions are used, are capable of being generalized with the help of generalized hypergeometric functions such as the generalized hypergeometeric series; MacRobert's E-function, Meijer's G- function. These generalizations may appear only mathematically interesting, and not appear to give meaningful physical interpretations at present. Nonetheless, it is expected that extensive study of the generalized hypergeometric functions together with the development of computers in future will open new frontiers of their applications which are hitherto unknown. However, some particular cases of our special solution may lead to meaningful physical interpretations.

The Meijer's [5] G-function of one variable was defined in terms of Mellin-Barnes type integrals as follows:

$$G_{p,q}^{m,n}\left[x\Big|_{(b_j,1)_{1,q}}^{(a_j,1)_{1,p}}\right] = \frac{1}{2\pi i}\int_L \theta(s)\, x^s ds \tag{1}$$

where i = √(– 1),

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}$$

and an empty product is interpreted as 1, $0 \le m \le q$, $0 \le n \le p$, and the parameters are such that no poles of $\Gamma(b_j - s)$ (j = 1, …, m) coincides with any pole of $\Gamma(1 - a_j + s)$ (j = 1, …, n).

There are three different paths L of integration:

(i) L runs from $-i\infty$ to $+i\infty$ so that all poles of $\Gamma(b_j - s)$ (j = 1, …, m) are to the right and all the poles of $\Gamma(1 - a_j + s)$ (j = 1, …, n) to the left of L. The integral converges if $p + q < 2(m + n)$ and $|\arg x| < (m + n - ½ p - ½ q)\pi$.

(ii) L is a loop starting and ending at $+\infty$ and encircling all poles of $\Gamma(b_j - s)$ (j = 1, …, m) once in the negative direction, but none of the poles of $\Gamma(1 - a_j + s)$ (j = 1, …, n). The integral converges if $q \ge 1$ and either $p < q$ or $p = q$ and $|x| < 1$.

(iii) L is a loop starting and ending at $-\infty$ and encircling all poles of $\Gamma(1 - a_j + s)$ (j = 1, …, n) once in the positive direction, but none of the poles of $\Gamma(b_j - s)$ (j = 1, …, m). The integral converges if $p \ge 1$ and either $p > q$ or $p = q$ and $|x| < 1$.

The following formulae are required in the proofs:

Bessel polynomial [4, p. 213, (2)]:

$$y_n(x; a, b) = \sum_{r=0}^{n} \frac{(-n)_r (a+n-1)_r}{r!} \left(-\frac{x}{b}\right)^r = {}_2F_0\left[{}^{-n, a+n-1;}_{-;} - x/b\right] \quad (2)$$

Orthogonality property of Bessel polynomials [4, p. 215, (14)]:

$$\int_0^\infty x^{a-2} e^{-\frac{1}{x}} y_m(x; a, 1) dx = \frac{(-1)^n n! (n+a-1)\pi}{\Gamma(a+n)\Gamma(2n+a-1)\sin \pi a} \delta_{m,n}, \quad (3)$$

where Re a</– m–n.

The Integral

$$\int_0^\infty x^{\sigma-1} e^{-\frac{1}{x}} y_n(x; a, 1) dx = \frac{\Gamma(-\sigma-n)\Gamma(a-\sigma-1+n)}{\Gamma(a-\sigma-1)} \quad (4)$$

where Re $\sigma < 0$, Re $(a - \sigma) < 2$, $\sigma \neq -1, -2, -3, \ldots$.

The integral (4) can easily established by expressing the Bessel Polynomial in the integrand as its series representation (2), interchainging the order of integration and summation, evaluating the resulting integral with the help of [2, p. 313(16)], using Gauss's theorem [1, p.104, (46)] and [1, p.3(3)]:

2. Integral:

The integral to be evaluated is

$$\int_0^\infty x^{\sigma-1}e^{-\frac{1}{x}}y_n(x;a,1)G_{p,q}^{u,v}[zx|_{(b_j,1)_{1,q}}^{(a_j,1)_{1,p}}]dx$$

$$= G_{p+1,q+2}^{u+2,v}\left[z|_{(-\sigma-n,1),(a-\sigma-1+n,1),(b_j,\beta_j)_{1,q}}^{(a_j,1)_{1,p},(a-\sigma-1,1)}\right]. \quad (5)$$

Proof:

To establish (5), expressing the G-function in the integrand as a Mellin-Barnes type integral (1) and interchainging the order of integrations which is justified due to the absolute convergence of the integrals involved in the process, evaluating the inner-integral with the help of (4) and using the definition of G-function, we arrive at (5).

3. Solution of the Time-Domain Synthesis Problem of Signals:

The classical time-domain synthesis problem occurring in electrical network theory is stated as follows [3, p.139]:

Given an electrical signal described by a real valued conventional function f(t) on $0 < t < \infty$, construct an electrical network consisting of finite number of components R, C and I which are all fixed, linear and positive, such that the output of $f_N(t)$, resulting from a delta-function $\delta(t)$ approximates f(t) on $0 < t < \infty$ in some sense.

In order to obtain a solution of this problem, we expand the function f(t) into a convergent series:

$$f(t) = \sum_{n=0}^{\infty} \psi_n(t) \quad (6)$$

of real-valued function $\psi_n(t)$. Let every partial sum

$$f_N(t) = \sum_{n=0}^{N} \psi_n(t),\ N = 0, 1, 2, \ldots \quad (7)$$

possess the two properties:

(i) $f_N(t) = 0$, for $-\infty < t < 0$.

(ii) The laplace transform $F_N(s)$of $f_N(t)$ is a rational function having a zero at $s = \infty$ and all its poles in the left-half s-plane, except possibly for a simple pole at the origin.

After choosing n in (7) sufficiently large to satisfy whatever approximation criterion is being used, an orthonormal series expansion may be employed. The Bessel polynomial transformation and (3) yields an immediate solution in the following form:

f(t) = $\sum_{n=0}^{\infty} c_n t^{(a-2)/2} e^{-1/2t} y_n(t; a, 1)$

where

$$c_n = (-1)^n \frac{\Gamma(a+n)\Gamma(2n+a-1)\sin\pi a}{n!(n+a-1)\pi} \int_0^{\infty} f(t) t^{(a-2)} e^{-1/2t} y_n(t; a, 1) dt \tag{8}$$

where Re a < 1 – 2n.

This case is an example of the use of hypergeometric integral in an orthonormal series expansion.

4. Particular Solution of the Problem:

The particular solution of the problem is

$$f(t) = \frac{sin\pi a}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(a+n)\Gamma(2n+a-1)}{n!(n+a-1)} t^{(a-2)/2} e^{-1/2t} y_n(t; a, 1)$$

$$\times G_{p+1,q+2}^{u+2,v}\left[z\Big|_{(-\sigma-n,1),(a-\sigma-1+n,1),(b_j,\beta_j)_{1,q}}^{(a_j,1)_{1,p},(a-\sigma-1,1)}\right]. \tag{9}$$

Proof:

Let us consider

f(t) = $t^{\sigma-a/2} e^{-1/2t} G_{p,q}^{u,v}[zx|_{(b_j,1)_{1,q}}^{(a_j,1)_{1,p}}]$

$$= \sum_{n=0}^{\infty} c_n t^{(a-2)/2} e^{-1/2t} y_n(t; a, 1) \tag{10}$$

Equation (10) is valid, since f(t) is continuous and of bounded variation in the open interval (0, ∞).

Multiplying both sides of (10) by $t^{(a-2)/2} e^{-1/2t} y_m(t; a, 1)$ and integrating with respect to t from 0 to ∞, we get

$$\int_0^{\infty} t^{\sigma-1} e^{-1/t} y_n(t; a, 1) G_{p,q}^{u,v}[zt|_{(b_j,1)_{1,q}}^{(a_j,1)_{1,p}}] dt$$

$$= \sum_{n=0}^{\infty} c_n \int_0^{\infty} t^{(a-2)} e^{-1/t} y_m(t; a, 1)\, dt$$

Now using (5) and (3), we get

$$c_m = \frac{(-1)^m \Gamma(a+m)(2m+a-1) sin\pi a}{m!(m+a-1)\pi} G^{u+2,v}_{p+1,q+2}\left[z\Big|^{(a_j,1)_{1,p},(a-\sigma-1,1)}_{(-\sigma-n,1),(a-\sigma-1+n,1),(b_j,\beta_j)_{1,q}}\right]$$

(11)

From (10) and (11), the solution (9) follows immediately.

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