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This Volume of

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is being dedicated to

Pingala

Acharya **Pingala** (*piṅgala*; c. 3rd–2nd century BCE) was an ancient Indian poet and mathematician, and the author of the *Chandaḥśāstra* (also called the *Pingala-sutras*), the earliest known treatise on Sanskrit prosody.

The *Chandaḥśāstra* is a work of eight chapters in the late Sūtra style, not fully comprehensible without a commentary. It has been dated to the last few centuries BCE. In the 10th century CE, Halayudha wrote a commentary elaborating on the *Chandaḥśāstra*. According to some historians Maharshi Pingala was the brother of Pāṇini the famous Sanskrit grammarian, considered the first descriptive linguist. Another think tank identifies him as Patanjali, the 2nd century CE scholar who authored Mahabhashya.

FOREWORD

The present volume of ***INSPIRE*** contains the various research papers of Faculty and Research Scholars of Department of Mathematics, INSTITUTE FOR EXCELLENCE IN HIGHER EDUCATION, BHOPAL (M. P.).

For me it is the realization of a dream which some of us have been nurturing for long and has now taken a concrete shape through the frantic efforts and good wishes of our dedicated band of research workers in our country, in the important area of mathematics.

The editor deserves to be congratulated for this very successful venture. The subject matter has been nicely and systematically presented and is expected to be of use to the workers.

(Dr. Pragyesh Kumar Agarwal)
Director & Patron
IEHE, Bhopal (M. P.)

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An Overview of Metric Spaces and Their Different Types

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Abstract

Metric spaces are mathematical structures that provide a framework for studying the concept of distance between points in a set. They have applications in various fields of mathematics, physics, computer science, and engineering. In this article, we provide an overview of metric spaces and their different types, including Quasi Metric Spaces, Pseudo Metric Spaces, Gahler's 2-Metric Space,Bi-metric Spaces, complete metric spaces, bounded metric spaces, compact metric spaces locally compact metric spacesand normed spaces. We discuss the properties and applications of each type of space, and present some open problems in the field.

Keywords

Metric space, complete metric space, bounded metric space, normed space, distance function, topology, completeness, etc.

Introduction

Metric spaces are a fundamental concept in mathematics that provide a framework for analyzing the concept of distance between points in a set. They have applications in various fields of science and engineering, including topology, analysis, geometry, physics, computer science, and optimization. In this article, we will discuss different types of metric spaces, their properties, and their applications.

Metric Spaces:A metric space is a mathematical structure that provides a way to measure distances between points. More formally, a metric space is a set X equipped with a metric d, which is a function that assigns a non-negative real number to every pair of points in X, such that the following conditions hold:

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangle inequality)

A classic example of a metric space is Euclidean space, where d(x, y) is the Euclidean distance between two points x and y. Metric spaces have many applications in mathematics and science, such as in optimization problems, data analysis, and computer science.

Quasi Metric Spaces: Quasi metric spaces are a generalization of metric spaces where the triangle inequality is replaced by a weaker condition. A quasi-metric space (X, d) is a set X equipped with a function $d : X \times X \to [0, \infty)$ such that:

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. $d(x, z) \leq Cd(x, y) + Cd(y, z)$ for all $x, y, z \in X$ and some constant $C \geq 1$.

In a quasi-metric space, the triangle inequality is replaced by a weaker condition that allows for a distortion factor. Quasi-metric spaces have been used in various areas of mathematics, such as analysis and topology.

Pseudo Metric Spaces: Pseudo metric spaces are another generalization of metric spaces that relax the conditions of symmetry and the triangle inequality. A pseudo-metric space (X, d) is a set X equipped with a function $d : X \times X \to [0, \infty)$ such that:

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The main difference between pseudo-metric spaces and metric spaces is that the triangle inequality is replaced by a weaker condition that allows for a reverse triangle inequality. Pseudo-metric spaces are used in various areas of mathematics, such as functional analysis and topology.

Gahler's 2-Metric Space: Gahler's 2-metric space is a modification of the standard metric space that incorporates two distinct metrics. A metric space is a set X along with a metric d that assigns a non-negative real number to each pair of points in X. Gahler's 2-metric space is a set X along with two metrics, d1 and d2, that assign non-negative real numbers to each pair of points in X.Formally, a Gahler's 2-metric space is defined as follows:

Let X be a non-empty set, and let d1 and d2 be two metrics on X. Then, (X, d_1, d_2) is a Gahler's 2-metric space if it satisfies the following properties:

1. For any two points x, y in X, $d_1(x, y) = 0$ if and only if $x = y$.
2. For any two points x, y in X, $d_2(x, y) = 0$ if and only if $x = y$.
3. For any three points x, y, z in X,
   $d_1(x, z) \leq d_1(x, y) + d_1(y, z)$ and
   $d_2(x, z) \leq d_2(x, y) + d_2(y, z)$ (the triangle inequality).

Gahler's 2-metric space has been studied extensively in the literature due to its applications in data analysis, computer science, and other fields. For example, Gahler's 2-metric space can be used to measure the similarity between two objects based on two different criteria. This technique has been applied in image analysis, bioinformatics, and machine learning.

Bi-metric Spaces: Bi-metric spaces are a generalization of Gahler's 2-metric space that involves more than two metrics. In a bi-metric space, a set X is equipped with multiple metrics, each of which provides a different notion of distance between points in X.

Formally, a bi-metric space is defined as follows:

Let X be a non-empty set, and let $d_1, d_2, ..., d_n$ be n metrics on X. Then, $(X, d_1, d_2, ..., d_n)$ is a bi-metric space if it satisfies the following properties:

1. For any two points x, y in X, $d_i(x, y) = 0$ if and only if x = y, for all i = 1, 2, ..., n.
2. For any three points x, y, z in X, $d_i(x, z) \leq di(x, y) + di(y, z)$ for all i = 1, 2, ..., n.

Bi-metric spaces have been studied in various fields, including computer science, physics, and engineering. These spaces have several applications, including image processing, pattern recognition, and clustering.

Bounded Metric Spaces:A bounded metric space is a space in which the distance between any two points is bounded by a fixed constant. These spaces have applications in functional analysis, optimization, and computer science. They are also important in the study of dynamical systems and their stability. The boundedness of metric spaces is related to the concept of Lipschitz continuity, which is an important topic in analysis.

Complete Metric Spaces:A complete metric space is a space in which every Cauchy sequence converges to a limit in the space. These spaces are important in the study of analysis, geometry, and physics. They have applications in the study of differential equations, probability theory, and optimization. The completeness of metric spaces is related to the concept of compactness, which is an important topic in topology.

Normed Spaces: A normed space is a vector space equipped with a norm, which is a function that assigns a non-negative value to each vector in the space. These spaces are important in functional analysis, where they provide a framework for studying linear operators and their properties. They have applications in the study of partial differential equations, quantum mechanics, and control theory.

Compact Metric Spaces: Compact metric spaces are those in which every open cover has a finite subcover. These spaces are important in topology and analysis, as they provide a framework for studying sequences and limits of functions. They have applications in differential equations, dynamical systems, and algebraic geometry.

Locally Compact Metric Spaces:A locally compact metric space is one in which every point has a compact neighborhood. These spaces have applications in probability theory, harmonic analysis, and number theory. They are also important in the study of Lie groups and their representations.

Conclusion

Metric spaces provide a powerful framework for studying the concept of distance between points in a set. There are many types of metric spaces, each with its own properties and applications. In this article, we have discussed several types of metric spaces, including compact metric spaces, locally compact metric spaces, complete metric spaces, and bounded metric spaces. We have also presented some of the main properties and applications of each type of space.

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Karush-Kuhn-Tucker Type Optimality Conditions and Duality in NonsmoothVector Minimization ProblemContaining Generalized Type-I Functions

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Abstract

A nonsmooth vector minimization problem involving generalized Type I vector valued functions are considerd in this paper. Here we have derived Karush-Kuhn-Tucker (KKT) type necessary and sufficient optimality conditions to get a solution which is efficient or properly efficient solution.Wolfe type and Mond-Weir type duals problems also have been considered under the generalized Type I assumptions in order to study other types of duality.

Key words: Duality, Generalized type I function, Nonsmoothvector minimization, Optimality.

1. Introduction

In nonlinear minimization problems, the convex functions play an important role, means convexity is frequently used hypotheses in optimization theory. Although convexity is not sufficient in many problems related to the fields of engineering, economics and many more. The notion of pseudoconvex and pseudoconcave functions as generalization of convex and concave functions were introduced in 1969 by Mangasarian [1]. Further in 1981 the concept of invex functions as a generalization of convexity for scalar constrained optimization problemswas introduced by Hanson [2]. Hanson came with some results on sufficiency of Kuhn-Tucker conditions for the scalar optimization problem.He also proved that weak duality and sufficiency of Kuhn-Tucker optimality conditions hold once invexity is needed rather than the usual requirement of convexity of the functions included in the problem. Hanson with his co-researchers introduced two new classes of functions type I and type II, which are not only sufficient but are also necessary for optimality in primal and dual problems[3].Pini et al. [4] analyzed some generalizations of convexity and their applications to duality theory and optimality conditions. Some more generalizations related to this field are produced by Rueda et al. [5], Zhao [6], Clarke [7], Kaul et al. [8], Rueda et al. [9], Suneja et al. [10] and Aghezzaf et al. [11].

By exploiting the above discussed and established results, in this present piece of work, we have considered anonsmooth vector minimization problem involving generalized Type I vector-valued functions which are defined in the manner considered by Clarke generalized gradient of locally Lipscitz functions and established KKT type necessary and sufficient optimality conditions.

Also we have established some Wolfe type and Mond-Weir type duality results for the nonsmooth vector minimization problem by using the above mentioned generalized Type I assumptions.

Present paper is divided in the sections. Section 2 includes some basic definitions and preliminaries. In section 3, KKT type necessary and sufficient conditions for a feasible solution to be an efficient solution or properly efficient have been established by us. Wherein the section 4, Wolfe type and Mond-Weir type duals under generalized Type I assumptions are provided. Lastly in section 5, we have concluded our results.

2. Basic Definitions and Preliminaries

Let R^k be the k-dimensional Euclidean space and R_+ be the positive orthant of R. Let S be any non-empty subset of R^k and $\langle .,. \rangle$ denote the Euclidean inner product. The following convention for equalities and inequalities will be adopted throughout the paper. If $x, y \in R^k$, we denote

$$x = y \iff x_i = y_i, \quad \forall i = 1,2,3,\dots,k;$$
$$x < y \iff x_i < y_i, \quad \forall i = 1,2,3,\dots,k;$$
$$x \leq y \iff x_i \leq y_i, \quad \forall i = 1,2,3,\dots,k \text{ but } x \neq y;$$
$$x \leqq y \iff x_i \leq y_i, \quad \forall i = 1,2,3,\dots,k.$$

Definition 2.1[1]A subset S of $R^k$is called convex if

$$\alpha u + (1-\alpha)v \in S, \qquad \forall\, u, v \in S, \qquad \forall\, \alpha \in [0,1]$$

Definition 2.2[1] Let S is any subset of R^k.The convex hull of S is the intersection of all convex sets in R^k containing S.

Definition 2.3[7] A function $g : S \to R$ is said to be locally Lipschitz at $w \in S$, if and only if there exists a positive number L and a neighborhood N of w such that, for any $y, z \in N$, one has

$|g(y) - g(z)| \leqq L\|y - z\|$.

The function g is said to be Lipschitz on S, if and only if the above condition is satisfied for all $w \in S$.

Definition 2.4[7] Let $g : S \to R$ be a locally Lipschitz function at $w \in S$. The Clarke generalized directional derivative of g at $w \in S$ in the direction of vector $v \in R^k$ is denoted by $g^o(w\,;v)$ and is defined as

$$g^o(w, v) = \limsup_{\substack{u \to w \\ t \downarrow 0}} \frac{g(u+tv) - g(u)}{t}.$$

Definition 2.5[7] Let $g : S \to R$ be a locally Lipschitz function at $w \in S$. The generalized gradient of g at $w \in S$ is denoted by $\partial g(w)$ and is defined as

$$\partial g(w) = \{\varphi \in R^k : \ g^o(w, v) \geq \varphi^T v, \ \forall\, v \in R^k\}.$$

We consider the following vector minimization problem (VMP)

$$\left.\begin{array}{r} minimize\ \ g(w), \\ subject\ to\ constraints\ \ h(w) \leqq 0 \end{array}\right\} \qquad (VMP)$$

where $g(w) = (g_1(w), g_2(w), g_3(w), \dots, g_m(w))$
$g_i: S \to R,\ i \in I = \{1, 2, 3, \dots, m\}$ and $h_j: S \to R,\ j \in J = \{1, 2, 3, \dots, n\}$ are locally Lipschitz functions on an open subset S of R^k.
Let $U = \{w \in S: h_j(w) \leqq 0, j \in J = \{1, 2, 3, \dots, n\}\}$.
For such vector minimization problems, the solution is defined by Sawaragi [12]in terms of a (properly) efficient solution.

Definition 2.6 A point $u \in U$ is said to be an efficient solution of the VMP, if and only if there exists no $w \in U$ such that

$$g(w) \leq g(u)$$

The set of all efficient solutions of the VMP be non-empty.

Definition 2.7 An efficient solution $u \in U$ is said to be properly efficient solution of the VMP, if and only if there exists a scalar $L > 0$ such that, for all $i \in I$ and for all $w \in U$ satisfying $g_i(w) < g_i(u)$, there exists at least one $q \in I$ such that $g_q(w) > g_q(u)$ and

$$\frac{g_i(u) - g_i(w)}{g_q(w) - g_q(u)} \leq L$$

Let g and h be locally Lipschitz functions at $u \in S$. We denote (g, h) as the pair of functions. The generalized Type I vector-valued functions are defined as below.

Definition 2.8 The pair$(g, h)$is called type I function with respect to θ at $u \in S$ if there exists a vector function $\theta(w, u)$defined on $U \times S$ such that, for all $w \in U$,

$$g_i(w) - g_i(u) \geqq \varphi_i^T\ \theta(w, u), \qquad \forall\ \varphi_i \in \partial g_i(u)$$
$$-h_j(u) \geqq \Psi_j^T\ \theta(w, u), \qquad \forall\ \Psi_j \in \partial h_j(u)$$

If $g_i(w) - g_i(u) > \varphi_i^T\ \theta(w, u)$, then (g, h) is said to be semistrictly-type I function with respect to θ at u.

Definition 2.9 The pair(g, h) is called quasi-type I function with respect to θ at $u \in S$ if there exists a vector function $\theta(w, u)$defined on $U \times S$ such that, for all $w \in U$,

$$g_i(w) \leqq g_i(u) \implies \varphi_i^T\ \theta(w, u) \leqq 0, \qquad \forall\ \varphi_i \in \partial g_i(u)$$
$$-h_j(u) \leqq 0 \implies \Psi_j^T\ \theta(w, u) \leqq 0, \qquad \forall\ \Psi_j \in \partial h_j(u)$$

Definition 2.10 The pair(g, h) is called pseudo-type I function with respect to θ at $u \in S$ if there exists a vector function $\theta(w, u)$defined on $U \times S$ such that, for all $w \in U$,

$$\varphi_i^T\ \theta(w, u) \geqq 0 \implies g_i(w) \geqq g_i(u), \qquad \forall\ \varphi_i \in \partial g_i(u)$$
$$\Psi_j^T\ \theta(w, u) \geqq 0 \implies -h_j(u) \geqq 0, \qquad \forall\ \Psi_j \in \partial h_j(u)$$

Definition 2.11 The pair(g, h) is called quasipseudo-type I function with respect to θ at $u \in S$ if there exists a vector function $\theta(w, u)$defined on $U \times S$ such that, for all $w \in U$,

$$g_i(w) \leqq g_i(u) \implies \varphi_i^T\,\theta(w,u) \leqq 0, \qquad \forall\, \varphi_i \in \partial g_i(u)$$
$$\Psi_j^T\,\theta(w,u) \geqq 0 \quad \Rightarrow -h_j(u) \geqq 0, \qquad \forall\, \Psi_j \in \partial h_j(u)$$

If the following inequality holds

$$\Psi_j^T\,\theta(w,u) \geqq 0 \quad \Rightarrow -h_j(u) > 0, \qquad \forall\, \Psi_j \in \partial h_j(u)$$

Then the pair (g,h) is called quasistrictly-pseudo-type I function at $u \in S$.

Definition 2.12 The pair(g,h) is called pseudoquasi-type I function with respect to θ at $u \in S$ if there exists a vector function $\theta(w,u)$defined on $U \times S$ such that, for all $w \in U$,

$$\varphi_i^T\,\theta(w,u) \geqq 0 \implies g_i(w) \geqq g_i(u), \qquad \forall\, \varphi_i \in \partial g_i(u)$$
$$-h_j(u) \leqq 0 \implies \Psi_j^T\,\theta(w,u) \leqq 0, \qquad \forall\, \Psi_j \in \partial h_j(u)$$

If the following inequality holds

$$\varphi_i^T\,\theta(w,u) \geqq 0 \implies g_i(w) > g_i(u), \qquad \forall\, \varphi_i \in \partial g_i(u)$$

Then the pair (g,h) is called strictly-pseudoquasi-type I function at $u \in S$.

3. OPTIMALITY CONDITIONS

Here in this section, we obtain KKT type necessary and sufficient conditions for a feasible solution u to be an efficient or properly efficient solution for VMP.

Let $J(u) = \{j \in J = \{1,2,3 \ldots ., n\}: h_j(u) = 0\}$

Theorem 3.1 Let u be a feasible solution for VMP and scalars $\lambda_i > 0, i = 1,2,3,\ldots,m$, $\mu_j \geqq 0, j = 1,2,3,\ldots,n$, $j \in J(u)$ such that

$$0 \in \sum_{i=1}^{m} \lambda_i\ \partial g_i(u) + \sum_{j\in J(u)} \mu_j\ \partial h_j(u) \tag{1}$$

If $(g,h_j)$is type I with respect to θ at u, then u is a properly efficient solution for VMP.

Proof: Since $(g,h_j)$is type I with respect to θ at u, by definition (**2.6**), for each $w \in U$, we have

$$\sum_{i=1}^{m} \lambda_i\, g_i(w) - \sum_{i=1}^{m} \lambda_i\, g_i(u) \geqq \sum_{i=1}^{m} \lambda_i\, \varphi_i^T\ \theta - (w,u), \ \ \forall\, \varphi_i \in \partial g_i(u) \tag{2}$$

$$0 = -\sum_{j\in J(u)} \mu_j\, h_j(u) \geqq \sum_{j\in J(u)} \mu_j\, \Psi_j^T\ \theta(w,u), \qquad \forall\, \Psi_j \in \partial h_j(u) \tag{3}$$

From condition (1), $\exists\ \varphi_i \in \partial g_i(u)$, $i = 1,2,3,\ldots,m$ and $\Psi_j \in \partial h_j(u), j \in J(u)$ such that

$$\sum_{i=1}^{m} \lambda_i\, \varphi_i + \sum_{j\in J(u)} \mu_j\, \Psi_j = 0 \tag{4}$$

On using (1), (2) and (3), we get

$$\sum_{i=1}^{m} \lambda_i\, g_i(w) - \sum_{i=1}^{m} \lambda_i\, g_i(u) \geqq 0$$

$$i.e. \quad \sum_{i=1}^{m} \lambda_i \, g_i(w) \geqq \sum_{i=1}^{m} \lambda_i \, g_i(u)$$

$i.e.$ u minimizes the following.

$$\sum_{i=1}^{m} \lambda_i \, g_i(w)$$

subject to $h(u) \leqq 0$

Therefore, u is a properly efficient solution for VMP due to theorem 1 of [13].

Theorem 3.2 Let u be a feasible solution for VMP. If there exist scalars $\lambda_i \geqq 0, i = 1, 2, 3, \dots, m,$ $\mu_j \geqq 0, j = 1, 2, 3, \dots, n,$ $j \in J(u)$ such that (1) of Theorem (3.1) holds and $(\lambda g, h_j)$, where $\lambda g = (\lambda_1 g_1, \lambda_2 g_2, \lambda_3 g_3, \dots, \lambda_m g_m)$, semistrictly-type I with respect to θ at u, then u is an efficient solution for VMP.

Proof: From (1) of theorem (3.1), we have

$$0 \in \sum_{i=1}^{m} \lambda_i \; \partial g_i(u) + \sum_{j \in J(u)} \mu_j \; \partial h_j(u)$$

$\exists \; \varphi_i \in \partial g_i(u), i = 1, 2, 3, \dots, m$ and $\Psi_j \in \partial h_j(u), j \in J(u)$ such that

$$\sum_{i=1}^{m} \lambda_i \, \varphi_i + \sum_{j \in J(u)} \mu_j \, \Psi_j = 0 \tag{1}$$

Let u is not an efficient solution for VMP, then there exists a feasible w for VMP and an index l such that

$g_l(w) < g_l(u),$
$g_p(w) \leqq g_p(u), \quad \forall \; l \neq p.$

Since $(\lambda g, h_j)$ is semistrictly-type Iwith respect to θ at u, we get

$$0 > \lambda_i \varphi_i^T \, \theta(w, u), \qquad \forall \; \varphi_i \in \partial g_i(u) \tag{2}$$

$$0 = -h_j(u) \geqq \Psi_j^T \, \theta(w, u), \qquad \forall \; \Psi_j \in \partial h_j(u), \qquad j \in J(u) \tag{3}$$

From (2) and (3), we get

$$\sum_{i=1}^{m} \lambda_i \varphi_i^T \, \theta(w, u) + \sum_{j \in J(u)} \mu_j \Psi_j^T \, \theta(w, u) < 0 \tag{4}$$

$$\forall \; \varphi_i \in \partial g_i(u), \quad \forall \; \Psi_j \in \partial h_j(u), \qquad j \in J(u)$$

Thus, inequality (4) contradicts (1).

Hence, u is an efficient solution for VMP.

Theorem 3.3 Let u be a feasible solution for VMP and there exist scalars $\lambda_i \geqq 0, i = 1, 2, 3, \dots, m, \sum_{i=1}^{m} \lambda_i = 1, \mu_j \geqq 0, j = 1, 2, 3, \dots, n, j \in J(u)$ such that (1) of theorem (3.1) holds. If $(\lambda g, \mu_j h_j)$, where $\lambda g = (\lambda_1 g_1, \lambda_2 g_2, \lambda_3 g_3, \dots, \lambda_m g_m)$, is pseudoquasi-type Iwith respect to θ at u, then u is a properly efficient solution for VMP.

Proof: Since $h_j(u) = 0, \mu_j \geqq 0$ and $(\lambda g, \mu_j h_j)$ is pseudoquasi-type Iwith respect to θ at u, therefore for all $w \in U$, we have

$$\Psi_j^T\ \theta(w,u) \leqq 0, \qquad \forall\, \Psi_j \in \partial h_j(u)$$

Using assumption (1) of theorem (3.1), $\exists\ \varphi_i \in \partial g_i(u),\ i = 1,2,3,\dots,m$ such that $\lambda_i \varphi_i^T\ \theta(w,u) \geqq 0,,\ \ i = 1,2,3,\dots,m,\ \forall\, w \in U$

Since $(\lambda g, \mu_j h_j)$ is pseudoquasi-type Iwith respect to θ at u, we get

$$\sum_{i=1}^{m} \lambda_i\, g_i(w) \geqq \sum_{i=1}^{m} \lambda_i\, g_i(u), \qquad \forall\, w \in U$$

Therefore, u minimizes $\sum_{i=1}^{m} \lambda_i\, g_i(w)$, subject to $h(w) \leqq 0$.

Hence, u is a properly efficient solution for VMP as in theorem 3.1.

Theorem 3.4 Let u be a feasible solution for VMP and there exist scalars $\lambda_i \geqq 0, i = 1,2,3,\dots,m, \sum_{i=1}^{m} \lambda_i = 1, \mu_j \geqq 0, j = 1,2,3,\dots,n, j \in J(u)$ such that (1) of theorem (3.1) holds. If $(\lambda g, \mu_j h_j)$, where $\lambda g = (\lambda_1 g_1, \lambda_2 g_2, \lambda_3 g_3, \dots, \lambda_m g_m)$, is strictly-pseudoquasi-type Iwith respect to θ at u, then u is an efficient solution for VMP.

Proof: From (1) of theorem (3.1), we have

$$0 \in \sum_{i=1}^{m} \lambda_i\ \partial g_i(u) + \sum_{j \in J(u)} \mu_j\ \partial h_j(u)$$

$\exists\ \varphi_i \in \partial g_i(u),\ i = 1,2,3,\dots,m$ and $\Psi_j \in \partial h_j(u), j \in J(u)$ such that

$$\sum_{i=1}^{m} \lambda_i\, \varphi_i + \sum_{j \in J(u)} \mu_j\, \Psi_j = 0 \qquad (1)$$

Let u is not an efficient solution for VMP, then there exists a feasible w for VMP and an index l such that

$g_l(w) < g_l(u)$,

$g_p(w) \leqq g_p(u),\ \ \forall\, l \neq p$.

Since $(\lambda g, \mu_j h_j)$ is strictly-pseudoquasi-type Iwith respect to θ at u and $h_j(u) = 0$, we get

$$\lambda_i \varphi_i^T\ \theta(w,u) < 0, \qquad \forall\, \varphi_i \in \partial g_i(u) \qquad (2)$$

$$\mu_j \Psi_j^T\ \theta(w,u) \leq 0, \qquad \forall\, \Psi_j \in \partial h_j(u), \qquad j \in J(u) \ (3)$$

From (2) and (3), we get

$$\sum_{i=1}^{m} \lambda_i \varphi_i^T\ \theta(w,u) + \sum_{j \in J(u)} \mu_j \Psi_j^T\ \theta(w,u) < 0 \qquad (4)$$

$$\forall\, \varphi_i \in \partial g_i(u), \quad \forall\, \Psi_j \in \partial h_j(u), \qquad j \in J(u)$$

Thus, inequality (4) contradicts (1). Hence, u is an efficient solution for VMP.

Definition 3.1 Let $g_i, i = 1,2,3,\dots,m$and $h_j, j = 1,2,3,\dots,n$ be locally Lipsticz functions at a point $u \in U$.

Problem VMP satisfies the Cottle constraint qualification at u if either $h_j(u) < 0$ for all $j = 1, 2, 3, \dots, n$or $0 \notin conv\{\partial h_j(u): h_j(u) = 0\}$, where $conv\ S$ denotes the convex hull of a set S.

Theorem 3.5 Assume that u is an efficient solution for VMP at which the Cottle constraint qualification is satisfied. Then there exist scalars $\lambda_i \geqq 0, i = 1, 2, 3, \dots, m$, $\sum_{i=1}^{m} \lambda_i = 1, \mu_j \geqq 0, j = 1, 2, 3, \dots, n$ such that

$$0 \in \sum_{i=1}^{m} \lambda_i \ \partial g_i(u) + \sum_{j=1}^{n} \mu_j \ \partial h_j(u),$$

$$\mu_j h_j(u) = 0, \qquad \forall\, j = 1, 2, 3, \dots, n$$

4. DUALITY

Wolfe type and Mond-Weir type duals under generalized Type Iassumption are taken in this section.

4.1 Wolfe Type Duality (WD):

We consider the following Wolfe [15] type dual for VMP.

(WD)$maximize \quad g(u) + \sum_{j=1}^{n} \mu_j \ \partial h_j(u) e,$ (1)

subject to the following:

$$0 \in \sum_{i=1}^{m} \lambda_i \ \partial g_i(u) + \sum_{j=1}^{n} \mu_j \ \partial h_j(u) \tag{2}$$

$$\lambda_i \geqq 0, \qquad i = 1, 2, 3, \dots, m \tag{3}$$

$$\sum_{i=1}^{m} \lambda_i = 1 \tag{4}$$

$$\mu_j \geqq 0, \qquad j = 1, 2, 3, \dots, n \tag{5}$$

where $e = (1, 1, 1, \dots, 1) \in R^m$.

Theorem 4.1.1 (Weak Duality) Let w be feasible for VMP and (u, λ, μ) feasible for WD. Assume that either (a) or (b) holds:

$(a)(g, h)$ is Type I at u with respect to θ and $\lambda > 0$;

$(b)(g, h)$ is semistrictly-Type I at u with respect to θ.

Then the following cannot hold:

$$g(w) \leq g(u) + \sum_{j=1}^{n} \mu_j \ \partial h_j(u) e$$

Proof: We prove the theorem by contradiction.

(a)Suppose the following holds.

$$g(w) \leq g(u) + \sum_{j=1}^{n} \mu_j \ \partial h_j(u) e$$

Since $\lambda > 0$, we find

$$\sum_{i=1}^{m} \lambda_i \, [g_i(w) - \{ \, g(u) + \sum_{j=1}^{n} \mu_j \, \partial h_j(u)\}] < 0 \tag{1}$$

Since (g, h) is Type I at u with respect to θ, (1) implies

$$\sum_{i=1}^{m} \lambda_i \varphi_i^T \, \theta(w,u) + \sum_{j=1}^{n} \mu_j \Psi_j^T \, \theta(w,u) < 0 \tag{2}$$

$$\forall \, \varphi_i \in \partial g_i(u), \quad \forall \, \Psi_j \in \partial h_j(u)$$

$\exists \;\; \varphi_i^* \in \partial g_i(u)$ and $\Psi_j^* \in \partial h_j(u)$ such that

$$\sum_{i=1}^{m} \lambda_i \, \varphi_i^* + \sum_{j=1}^{n} \mu_j \Psi_j^* = 0 \tag{3}$$

This implies that

$$\sum_{i=1}^{m} \lambda_i \varphi_i^* \, \theta(w,u) + \sum_{j=1}^{n} \mu_j \Psi_j^* \, \theta(w,u) = 0 \tag{4}$$

which contradicts (2). This completes the proof.

(b)Suppose the following holds.

$$g(w) \leq g(u) + \sum_{j=1}^{n} \mu_j \, \partial h_j(u) e$$

We find the following inequality.

$$\sum_{i=1}^{m} \lambda_i \, [g_i(w) - \{ \, g(u) + \sum_{j=1}^{n} \mu_j \, \partial h_j(u)\}] \leqq 0 \tag{5}$$

Since (g, h) is semistrictly-Type I at u with respect to θ, (5) implies

$$\sum_{i=1}^{m} \lambda_i \varphi_i^T \, \theta(w,u) + \sum_{j=1}^{n} \mu_j \Psi_j^T \, \theta(w,u) < 0 \tag{6}$$

$$\forall \, \varphi_i \in \partial g_i(u), \quad \forall \, \Psi_j \in \partial h_j(u)$$

$\exists \;\; \varphi_i^* \in \partial g_i(u)$ and $\Psi_j^* \in \partial h_j(u)$ such that

$$\sum_{i=1}^{m} \lambda_i \, \varphi_i^* + \sum_{j=1}^{n} \mu_j \Psi_j^* = 0 \tag{7}$$

This implies that

$$\sum_{i=1}^{m} \lambda_i \varphi_i^* \, \theta(w,u) + \sum_{j=1}^{n} \mu_j \Psi_j^* \, \theta(w,u) = 0 \tag{8}$$

which contradicts (6). This completes the proof.

Theorem 4.1.2 (Strong Duality) Let $\overline{w}$ be an efficient solution for VMP at which the Cottle constraint qualification is satisfied. Then there exist $\bar{\lambda} \in R^m$ and $\bar{\mu} \in R^n$ such that $(\overline{w}, \bar{\lambda}, \bar{\mu})$ is feasible for WD.

If also (g, h) is semistrictly-Type I with respect to θat $u \in U$, then $(\bar{w}, \bar{\lambda}, \bar{\mu})$ is an efficient solution for WD.

Proof: Since $\bar{w}$ is an efficient solution for VMP at which the Cottle constraint qualification is satisfied at $\bar{w}$, from theorem 3.5 , there exist scalars $\bar{\lambda}_i \geqq 0, i = 1, 2, 3, \ldots, m$, $\sum_{i=1}^{m} \bar{\lambda}_i = 1, \bar{\mu}_j \geqq 0, j = 1, 2, 3, \ldots, n$ such that (1) and (2) of theorem 3.5 hold.

Hence $(\bar{w}, \bar{\lambda}, \bar{\mu})$ is feasible for WD.

If $(\bar{w}, \bar{\lambda}, \bar{\mu})$ is not an efficient solution for WD, then there exists a feasible solution (u, λ, μ) for WD such that

$$\{g(\bar{w}) + \sum_{j=1}^{n} \bar{\mu}_j \partial h_j(\bar{w}) e\} \leq \{g(u) + \sum_{j=1}^{n} \mu_j \, \partial h_j(u) e\}$$

which contradicts part (b) of theorem 4.1.1 for feasible solution $\bar{w}$ for VMP and (u, λ, μ) for WD.

Hence $(\bar{w}, \bar{\lambda}, \bar{\mu})$ is an efficient solution for WD.

4.2 Mond-WeirType Duality (MWD):

We consider the following Mond-Weir [16] type dual for problem VMP:

(MWD)$maximize \quad g(u),$ (1)

subject to the following:

$$0 \in \sum_{i=1}^{m} \lambda_i \; \partial g_i(u) + \sum_{j=1}^{n} \mu_j \; \partial h_j(u) \tag{2}$$

$$\lambda_i \geqq 0, \qquad i = 1, 2, 3, \ldots, m \tag{3}$$

$$\sum_{i=1}^{m} \lambda_i = 1 \tag{4}$$

$$\mu_j h_j(u) \geqq 0, \qquad j = 1, 2, 3, \ldots, n \tag{5}$$

$$\mu_j \geqq 0, \qquad j = 1, 2, 3, \ldots, n \tag{6}$$

Theorem 4.2.1 (Weak Duality) Let w be feasible for VMP and (u, λ, μ) feasible for MWD. If $(\lambda g, \mu h)$ is quasistrictly-pseudo-Type I at u with respect to θ, then

$$g(w) \nleq g(u)$$

Proof : We prove the theorem by contradiction.

Suppose the following holds.

$$g(w) \leq g(u)$$

then there exists an index l such that

$g_l(w) < g_l(u), g_p(w) \leqq g_p(u), \quad \forall \, l \neq p.$

Since $(\lambda g, \mu h)$ is quasistrictly-pseudo-Type I at u with respect to θ, the above inequalities and condition (5) of (4.2) imply

$$\sum_{i=1}^{m} \lambda_i \varphi_i^T \, \theta(w, u) \leqq 0, \qquad \forall \, \varphi_i \in \partial g_i(u) (1)$$

and

$$\sum_{j=1}^{n} \mu_j \Psi_j^T \, \theta(w,u) < 0, \quad \forall\, \Psi_j \in \partial h_j(u) \tag{2}$$

From (1) and (2), we get

$$\sum_{i=1}^{m} \lambda_i \varphi_i^T \, \theta(w,u) + \sum_{j=1}^{n} \mu_j \Psi_j^T \, \theta(w,u) < 0 \tag{3}$$

$$\forall\, \varphi_i \in \partial g_i(u), \quad \forall\, \Psi_j \in \partial h_j(u)$$

From condition (2), $\exists\ \varphi_i^* \in \partial g_i(u)$ and $\Psi_j^* \in \partial h_j(u)$ such that

$$\sum_{i=1}^{m} \lambda_i \, \varphi_i^* + \sum_{j=1}^{n} \mu_j \Psi_j^* = 0 \tag{4}$$

This implies that

$$\sum_{i=1}^{m} \lambda_i \varphi_i^* \, \theta(w,u) + \sum_{j=1}^{n} \mu_j \Psi_j^* \, \theta(w,u) = 0 \tag{5}$$

which contradicts (3).

Hence the following cannot hold:

$$g(w) \le g(u)$$

Theorem 4.2.2 (Strict Converse Duality) Let $\bar{w}$ be feasible solution for VMP and let $(\bar{w}, \bar{\lambda}, \bar{\mu})$ be feasible for MWD such that

$$\sum_{i=1}^{m} \bar{\lambda}_i \, g(\bar{w}) \leqq \sum_{i=1}^{m} \bar{\lambda}_i \, g(\bar{u}), \quad i = 1, 2, 3, \dots, m \tag{1}$$

If $(\bar{\lambda} g, \bar{\mu} h)$ is quasistrictly-pseudo-Type I at $\bar{u}$ with respect to θ, then $\bar{w} = \bar{u}$.

Proof: We prove the theorem by contradiction.

Let $\bar{w} \neq \bar{u}$

Since $(\bar{w}, \bar{\lambda}, \bar{\mu})$ is feasible for MWD, then from condition (2)of (4.2) there exist $\exists\ \varphi_i \in \partial g_i(\bar{u}),\ i = 1, 2, 3, \dots, m$ and $\Psi_j \in \partial h_j(\bar{u}),\ j = 1, 2, 3, \dots, n$ such that

$$\sum_{i=1}^{m} \bar{\lambda}_i \, \varphi_i + \sum_{j=1}^{n} \bar{\mu}_j \, \Psi_j = 0 \tag{2}$$

Since $(\bar{\lambda} g, \bar{\mu} h)$ is quasistrictly-pseudo-Type I at $\bar{u}$ with respect to θ, we find

$$\bar{\lambda}_i g_i(\bar{w}) \leqq \bar{\lambda}_i g_i(\bar{u}) \Rightarrow \bar{\lambda}_i \varphi_i^T \, \theta(\bar{w}, \bar{u}) \leqq 0, \quad \forall\, \varphi_i \in \partial g_i(\bar{u}) \tag{3}$$

and

$-\bar{\mu}_j h_j(\bar{u}) \leqq 0 \Rightarrow \bar{\mu}_j \Psi_i^T \, \theta(\bar{w}, \bar{u}) < 0, \quad \forall\, \Psi_i \in \partial h_j(\bar{u})(4)$

Combining (1) and (3), we have

$$\bar{\lambda}_i \varphi_i^T \, \theta(\bar{w}, \bar{u}) \leqq 0, \quad \forall\, \varphi_i \in \partial g_i(\bar{u}),$$

which implies that

$$\textstyle\sum_{i=1}^{m} \bar{\lambda}_i \varphi_i^T \, \theta(\bar{w}, \bar{u}) \leqq 0, \quad \forall\, \varphi_i \in \partial g_i(\bar{u}) \tag{5}$$

Since $(\bar{w},\bar{\lambda},\bar{\mu})$ is feasible for MWD, we have

$$-\bar{\mu}_j h_j(\bar{u}) \leqq 0, \quad j = 1,2,3,\dots,n$$

Using (4) in above inequality, we get

$$\bar{\mu}_j \Psi_i^T\, \theta(\bar{w},\bar{u}) < 0, \qquad j = 1,2,3,\dots,n$$

which implies that

$$\sum_{j=1}^{n} \bar{\mu}_j \Psi_j^T\, \theta(\bar{w},\bar{u}) < 0 \tag{6}$$

Combining (5) and (6), we get

$$\sum_{i=1}^{m} \bar{\lambda}_i \varphi_i^T\, \theta(\bar{w},\bar{u}) + \sum_{j=1}^{n} \bar{\mu}_j \Psi_j^T\, \theta(\bar{w},\bar{u}) < 0 \tag{7}$$

which contradicts (2).
Hence $\bar{w} = \bar{u}$.

5. CONCLUSIONS:

Finally we claim that we have obtained KKT type necessary and sufficient conditions for a feasible solution to be an efficient solution or properly efficient solution. Also we have proved different types of duality theorems for Wolfe type and Mond-Weir type duals under the generalized Type I assumptions.

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Some Integrals as the Product of M-Series and I-function

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Abstract

Here we have established some new finite integrals involving product of hypergeometric functions, generalized hypergeometric functions with exponential functions also we have found and studied some new finite integrals involving the product of M-Series and I-function with exponential functions , which are more generalized in nature and able to produce many new results. Inayat Hussain [5] introduced that The M-series is a particular case of $\bar{H}$-function. In 2008 Saxena V.P. [16] gave the concept of I-function and M-Series which are strongly helpful to solve the problems of various fields' especially mathematical physics, biology, data mining signal and image processing and many more.

Keywords: Exponential function, Fox H-function, $\bar{H}$-function, hypergeometric function, Saxena's I-function, M-series, Mellin-Barnes type integral.

1. Introduction

Many researchers Heine, Goursat, Pochhamar, Appell and others generalized Gauss hypergeomtric function in their own ways. Generalized ordinary hypergeometric series is defined in the following manner:

$$ {}_pF_q[(a_p);(b_q);z] = {}_pF_q\left[{(a_p); \atop (b_q);} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{z^n}{n!}, \tag{1.1} $$

where, for brevity, (a_p) denotes the array of p parameters $a_1, \cdots, a_p$ with similar interpretation for (b_q) etc. ${}_pF_q$ is not defined if any denominator parameter b_q is a negative integer or zero . In 1937-38 Mac Robert introduced a function known as Mac Robert's E-function [14], explaining the meaning of ${}_pF_q$ function even when $p > q + 1$.In his dedicated work [10,11,12] Meijer gave the concept of G-function as a sum of certain ${}_pF_q$ functions. Now G-function is a generalization of higher transcendental function given in [1].

In order to get the solutions of some functional equations H-function occurs, which was studied by many researchers such as Bochner [2] ,Chandrashekhran and Narsimhan [3].Further [6,8,9,21,23] extended the concept and generalized the results existing already .

Fox [4] has defined the H-function in terms of a general Mellin–Barnes type integral as follows:

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_j\,,\,\alpha_j)_{1,p}\\(b_j\,,\,\beta_j)_{1,q}\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_{\mathcal{L}}\theta(s)x^s ds, \tag{1.2}$$

where $\omega=\sqrt{-1}\,, x(\neq 0)$ is a complex variable and x^s = $\exp[s\,\{\log|x|+\omega\arg x\}]$ in which $\log|x|$ represents the natural logarithm of $|x|$ and $arg\ x$ is not necessarily the principal value. An empty product is interpreted as unity. Also,

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\ \prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\ \prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)} \tag{1.3}$$

The I-function which was produced by Saxena [15] in 1982 , further studied by many author's G. D. Vaishya, R. Jain and R. C. Verma [22], C. K. Sharma [19] etc.

In this present piece of work, we are defining and representing the I-function as follow:

$$I_{p_i,q_i:r}^{m,n}[z]=I_{p_i,q_i:r}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\ \phi(\xi)z^{\xi}d\xi\ , \tag{1.4}$$

where

$$\phi(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\ \prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\ \prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}}, \tag{1.5}$$

$p_i(i=1,2,\dots,r), q_i(i=1,2,\dots,r), m, n$ are integers satisfying $0\leq n\leq p_i, 0\leq m\leq q_i$ $(i=1,2,\dots,r)$; r is finite, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}$ are real and positive; a_j,b_j,a_{ji},b_{ji} are complex numbers and $\mathcal{L}$ is the path of integration separating the increasing and decreasing sequences of poles of the integrand and the convergence, existence conditions and other details of the I-function, one can refer to [16, p. 26-27]. The integral converges, if $|arg\,x|<\frac{1}{2}\pi\Omega_i$, where

$$\Omega_i=\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p_i}\alpha_j+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q_i}\beta_j>0$$

$$and\ T=\sum_{j=1}^{q_i}b_j-\sum_{j=1}^{p_i}a_j>0 \tag{1.6}$$

Recently Sharma Manoj introduce the M- series and discussed their properties. The M series is a generalization of Hypergeometric function [21], the M-Series is a particular case of the $\bar{H}$- function of Inayat-Hussain [5] and A special role in the application of fractional calculus operators and in the solution of fractional order differential equations. The hypergeometric function and Mittag-Laffer function follow as its particular case of [7], [17]. Therefore, it is very interesting. We defined by means of the following series expansion:

$$ {}_PM_Q^{\tau}\left(a_1 \dots . a_p; , b_1 \dots . b_q; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{\Gamma(\tau k+1)} \qquad (1.7) $$

Here $\mathrm{R}(\tau) > 0, (\mathrm{a_j})_{\mathrm{k}}$ $(\mathrm{b_j})_{\mathrm{k}}$ are pochammer symbols. For convergence conditions and other details of the generalized M -series see Sharma and Jain [18].

In the next section we will discuss the existing results, lemma and theorems which are necessary establish our main results.

2. Existing Works

The well-known Mellin inversion theorem which is known as Mellin transformation of the I- function may be given as

$$ \int_0^{\infty} x^{-s} I_{p_i,q_i:r}^{m,n} \left[z \left| \begin{matrix} (a_j,\alpha_j)_{1,n}; (a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m}; (b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] dx = a^{-s}\theta(-s) $$

$$ = a^{-s} \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j \xi) \ \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j \xi)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} \xi) \ \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} \xi) \right\}} \qquad (2.1) $$

integral converges, if $|arg\, x| < \frac{1}{2}\pi\Omega_i$, and follow the conditions (1.3) and

$$ - \min_{1\le j\le m} [Re(\frac{b_j}{\beta_j})] < Re(s) < \min_{1\le j\le n} [Re\left\{\frac{1-a_j}{\alpha_j}, \right\}] $$

Lemma 2.1: From the E.D.Rainville [13]. We have

$$ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k) \qquad (2.2) $$

4. Main Results

Here we have established the following integrals in the form of composition of exponential function, I-function and M- series.

Integral- I

$$ \int_0^t x^{\rho-1} (t-x)^{\sigma-1} e^{-xy} \ {}_PM_Q^{\tau} \left[(g_P); (h_Q); a x^{\gamma} (t-x)^{\delta} \right] $$

$$ \bullet I_{p_i,q_i:r}^{m,n} \left[z x^{\mu} (t-x)^{\nu} \left| \begin{matrix} (a_j,\alpha_j)_{1,n}; (a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m}; (b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] dx $$

$$= e^{-ty} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} t^{(\gamma+\delta-1)k+u} \frac{y^{u-k}}{(u-k)!} f(k)\, I^{m,n+2}_{p_i+2,q_i+1:r}\left[zt^{\mu+\nu}\left|\begin{matrix}P_1\\Q_1\end{matrix}\right.\right], \tag{3.1}$$

where P_1 and Q_1 denotes the parameter

$$(1-\rho-\gamma k,\mu),(1-\sigma-(\delta-1)k-u,\nu),\left(a_j,\alpha_j\right)_{1,n};\left(a_{ji},\alpha_{ji}\right)_{n+1,p_i}$$

and

$$\left(b_j,\beta_j\right)_{1,m};\left(b_{ji},\beta_{ji}\right)_{m+1,q_i},(1-\rho-\sigma-(\gamma+\delta-1)k-u-\delta k,\mu+\nu)$$

respectively. Also

$$f(k) = \frac{(g_1)_k \dots \left(g_p\right)_k}{(h_1)_k \dots \left(h_p\right)_k} \frac{a^k}{\Gamma(\tau k+1)}. \tag{3.2}$$

The conditions of validity of the integral (3.1) are

a. $Re(\rho) + \mu \min_{1\le j\le m}\left[Re\left(b_j/\beta_j\right)\right] > 0$ and $Re(\sigma) + \nu \min_{1\le j\le m}\left[Re\left(b_j/\beta_j\right)\right] > 0$
b. $Re(\mu) \ge 0, Re(\nu) \ge 0$, (not both zero simultaneously),
c. $\gamma,\delta,$ are non-negative integers, such that $\gamma+\delta \ge 1$, $\tau \in C, Re(\tau) > 0$, $T > 0, \Omega_i > 0, |\arg z| < \frac{1}{2}\pi\Omega_i$, where

$$\Omega_i = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p_i} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q_i} \beta_j > 0 \quad and$$

$$T = \sum_{j=1}^{q_i} b_j - \sum_{j=1}^{p_i} a_j > 0.$$

Integral- II

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-xy}\, {}_P M_Q^{\tau}\left[(g_P);\left(h_Q\right);ax^{\gamma}(t-x)^{\delta}\right]$$

$$\bullet I^{m,n}_{p_i,q_i:r}\left[zx^{-\mu}(t-x)^{-\nu}\left|\begin{matrix}\left(a_j,\alpha_j\right)_{1,n};\left(a_{ji},\alpha_{ji}\right)_{n+1,p_i}\\ \left(b_j,\beta_j\right)_{1,m};\left(b_{ji},\beta_{ji}\right)_{m+1,q_i}\end{matrix}\right.\right]dx$$

$$= e^{-ty} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} t^{(\gamma+\delta-1)k+u} \frac{y^{u-k}}{(u-k)!} f(k)\, I^{m+2,n}_{p_i+1,q_i+2:r}\left[zt^{-\mu-\nu}\left|\begin{matrix}P_2\\Q_2\end{matrix}\right.\right], \tag{3.3}$$

provided that

$Re(\rho) - \mu \min_{1\le j\le n}\left[Re\left((a_j-1)/\alpha_j\right)\right] > 0;\; Re(\sigma) - \nu \min_{1\le j\le n}\left[Re\left((a_j-1)/\alpha_j\right)\right] > 0$

and the sets of conditions (b) to (c) given with (4) are satisfied [$f(k)$ is given by (3.2)]. Here

$$P_2 = \left(a_j, \alpha_j\right)_{1,n}; \left(a_{ji}, \alpha_{ji}\right)_{n+1,p_i}, (\rho + \sigma + (\delta + \gamma - 1)k + u, \mu + \nu)$$
$$Q_2 = (\rho + \gamma k, \mu), (\sigma + (\delta - 1)k + u, \nu), \left(b_j, \beta_j\right)_{1,m}; \left(b_{ji}, \beta_{ji}\right)_{m+1,q_i}.$$

Integral- III

$$\int_0^t x^{\rho-1} (t-x)^{\sigma-1} e^{-xy} \, {}_P M_Q^{\tau}\left[(g_P); (h_Q); a x^{\gamma} (t-x)^{\delta}\right]$$
$$\bullet I_{p_i,q_i:r}^{m,n}\left[z x^{\mu} (t-x)^{-\nu} \left| \begin{matrix} \left(a_j, \alpha_j\right)_{1,n}; \left(a_{ji}, \alpha_{ji}\right)_{n+1,p_i} \\ \left(b_j, \beta_j\right)_{1,m}; \left(b_{ji}, \beta_{ji}\right)_{m+1,q_i} \end{matrix} \right.\right] dx$$
$$= e^{-ty} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} t^{(\gamma+\delta-1)k+u} \frac{y^{u-k}}{(u-k)!} f(k) \, I_{p_i+1,q_i+2:r}^{m+1,n+1}\left[z t^{\mu-\nu} \left| \begin{matrix} P_3 \\ Q_3 \end{matrix} \right.\right] \tag{3.4}$$

provided that $\mu > 0,\ \nu \geq 0$ such that $\mu - \nu \geq 0$ and
$Re(\rho) + \mu \min_{1\leq j\leq m}\left[Re\left(b_j/\beta_j\right)\right] > 0 \quad Re(\sigma) - \nu \min_{1\leq j\leq n}\left[Re\left((a_j - 1)/\alpha_j\right)\right] > 0,$ it being assumed that the conditions (b) to (c) given with integral I are satisfied. Also f(k) is given by (3.2).

Integral- IV

$$\int_0^t x^{\rho-1} (t-x)^{\sigma-1} e^{-xy} \, {}_P M_Q^{\tau}\left[(g_P); (h_Q); a x^{\gamma} (t-x)^{\delta}\right]$$
$$\bullet I_{p_i,q_i:r}^{m,n}\left[z x^{\mu} (t-x)^{-\nu} \left| \begin{matrix} \left(a_j, \alpha_j\right)_{1,n}; \left(a_{ji}, \alpha_{ji}\right)_{n+1,p_i} \\ \left(b_j, \beta_j\right)_{1,m}; \left(b_{ji}, \beta_{ji}\right)_{m+1,q_i} \end{matrix} \right.\right] dx$$
$$= e^{-ty} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} t^{(\gamma+\delta-1)k+u} \frac{y^{u-k}}{(u-k)!} f(k) \, I_{p_i+2,q_i+1:r}^{m+1,n+1}\left[z t^{\nu-\mu} \left| \begin{matrix} P_4 \\ Q_4 \end{matrix} \right.\right], \tag{3.5}$$

provided that $\mu \geq 0,\ \nu > 0$ such that $\nu - \mu \geq 0$;
$Re(\rho) - \mu \min_{1\leq j\leq n}\left[Re\left((a_j - 1)/\alpha_j\right)\right] > 0 \ Re(\sigma) + \nu \min_{1\leq j\leq m}\left[Re\left(b_j/\beta_j\right)\right] > 0.$

Here

$$P_4 = (1 - \rho - \gamma k, \mu), \left(a_j, \alpha_j\right)_{1,n}; \left(a_{ji}, \alpha_{ji}\right)_{n+1,p_i}, (\rho + \sigma + (\delta + \gamma - 1)k + u, \nu - \mu)$$
$$Q_4 = (\sigma + (\delta - 1)k + u, \nu), \left(b_j, \beta_j\right)_{1,m}; \left(b_{ji}, \beta_{ji}\right)_{m+1,q_i}.$$

and f(k) is given by (3.2).

Integral- V

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-xy}\ {}_PM_Q^{\tau}\left[(g_P);(h_Q);ax^{\gamma}(t-x)^{\delta}\right]$$
$$\bullet I_{p_i,q_i:r}^{m,n}\left[zx^{-\mu}(t-x)^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dx$$
$$= e^{-ty}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}t^{(\gamma+\delta-1)k+u}\ \frac{y^{u-k}}{(u-k)!}\ f(k)\,I_{p_i+2,q_i+1:r}^{m+1,n+1}\left[zt^{\nu-\mu}\left|\begin{matrix}P_4\\Q_4\end{matrix}\right.\right], \quad (3.6)$$

provided that $\mu \geq 0,\ \nu > 0$ such that $\nu-\mu \geq 0$;
$Re(\rho)+\mu \min_{1\le j\le n}\left[Re\left(\frac{b_j}{\beta_j}\right)\right] > 0\ Re(\sigma)-\nu \max_{1\le j\le m}\left[Re\left((a_j-1)/\alpha_j\right)\right] > 0.$
Here

$$P_4 = (1-\rho-(\gamma-1)k-u,\nu),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i},$$
$$Q_4 = (1-\sigma-\delta k,\nu),(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}.$$

and f(k) is defined by (3.2).

Integral- VI

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-xy}\ {}_PM_Q^{\tau}\left[(g_P);(h_Q);ax^{\gamma}(t-x)^{\delta}\right]$$
$$\bullet I_{p_i,q_i:r}^{m,n}\left[zx^{-\mu}(t-x)^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dx$$
$$= e^{-ty}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}t^{(\gamma+\delta-1)k+u}\ \frac{y^{u-k}}{(u-k)!}\ f(k)\,I_{p_i+2,q_i+1:r}^{m+1,n+1}\left[zt^{\nu-\mu}\left|\begin{matrix}P_4\\Q_4\end{matrix}\right.\right], \quad (3.7)$$

provided that $\mu \geq 0,\ \nu > 0$ such that $\nu-\mu \geq 0$;
$Re(\rho)-\mu \min_{1\le j\le n}\left[Re\left((a_j-1)/\alpha_j\right)\right] > 0\ Re(\sigma)+\nu \min_{1\le j\le m}\left[Re\left(b_j/\beta_j\right)\right] > 0.$ Here

$$P_4 = (\rho+\gamma k,\mu),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i},(\rho+\sigma+\gamma k+\delta k,\nu-\mu)$$
$$Q_4 = (\rho+\gamma k,\mu),(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}(1-\sigma-(\delta+\gamma-1)k-u,\nu-\mu).$$

and f(k) is defined by (3.2).

Proof of the Integrals:

In order to prove first integral (3.1), let us consider L.H.S. of (3.1)
:

$$I_1 = e^{-ty}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{(t-x)y}\ {}_PM_Q^{\tau}\left[(g_P);(h_Q);ax^{\gamma}(t-x)^{\delta}\right]$$
$$\bullet I_{p_i,q_i:r}^{m,n}\left[zx^{\mu}(t-x)^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dx$$

Now we replace $e^{(t-x)y}$ by $\sum_{u=0}^{\infty}\frac{(t-x)^u y^u}{u!}$ and express the M-series and the I-function with the help of (1.7) and (1.4) respectively, then we get

$$I_1 = e^{-ty}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\frac{(t-x)^u y^u}{u!}\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{a^k x^{\gamma k}(t-x)^{\delta k}}{\Gamma(\tau k+1)}$$
$$\bullet\frac{1}{2\pi i}\int_{\mathcal{L}}\phi(\xi)z^{\xi}x^{\mu\xi}(t-x)^{\nu\xi}d\xi dx$$
$$= e^{-ty}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{a^k x^{\gamma k}(t-x)^{\delta k+u}}{\Gamma(\tau k+1)}\frac{y^u}{u!}$$
$$\bullet\frac{1}{2\pi i}\int_{\mathcal{L}}\phi(\xi)z^{\xi}x^{\mu\xi}(t-x)^{\nu\xi}d\xi dx$$

Now by the use of (2.2), the above result reduces to

$$= e^{-ty}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{a^k x^{\gamma k}(t-x)^{\delta k+u-k}}{\Gamma(\tau k+1)}\frac{y^{u-k}}{(u-k)!}$$
$$\bullet\frac{1}{2\pi i}\int_{\mathcal{L}}\phi(\xi)z^{\xi}x^{\mu\xi}(t-x)^{\nu\xi}d\xi dx$$

Interchanging the order of integration and summation, we get

$$I_1 = e^{-ty}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{y^{u-k}}{(u-k)!}\frac{1}{2\pi i}\int_{\mathcal{L}}\phi(\xi)z^{\xi}$$
$$\bullet\left\{\int_0^t x^{\rho+\gamma k+\mu\xi-1}(t-x)^{\sigma+(\eta-1)k+u+\nu\xi-1}dx\right\}d\xi,$$

where f(k) is given by (3.2).

on substituting x= ts in the inner x-integral, the above expression reduce to

$$I_1 = e^{-ty}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{y^{u-k}}{(u-k)!}t^{(\delta+\eta-1)k+u}\frac{1}{2\pi i}\int_{\mathcal{L}}\phi(\xi)z^{\xi}t^{(\mu+\nu)\xi}$$

$$\bullet \left\{ \int_0^1 s^{\rho+\gamma k+\mu\xi-1} (1-s)^{\sigma+(\eta-1)k+u+\nu\xi-1} ds \right\} d\xi,$$

$$= e^{-ty} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{y^{u-k}}{(u-k)!} t^{(\delta+\eta-1)k+u} \frac{1}{2\pi i} \int_{\mathcal{L}} \phi(\xi)$$

$$\bullet \frac{\Gamma(\rho+\delta k+\mu\xi)\Gamma(\sigma+(\eta-1)k+u+\nu\xi)}{\Gamma(\rho+\sigma+(\delta+\eta-1)k+u+(\mu+\nu)\xi} z^{\xi} t^{(\mu+\nu)\xi} \, d\xi$$

Finally, interpreting the contour integral by virtue of (1.5), we obtain

$$I_1 = e^{-yt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{y^{u-k}}{(u-k)!} t^{(\delta+\eta-1)k+u}$$

$$\bullet I^{m,n+2}_{p_i+2,q_i+1:r} \left[z t^{\mu+\nu} \middle| \begin{matrix} P_1 \\ Q_1 \end{matrix} \right]$$

where P_1 and Q_1 denotes the parameter

$$P_1 = (1-\rho-\gamma k,\mu),(1-\sigma-(\delta-1)k-u,\nu),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}$$

and

$$Q_1 = (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(1-\rho-\sigma-(\gamma+\delta-1)k-u-\delta k,\mu+\nu)$$

By using similar mathematical analysis we can prove the remaining integrals

Special Cases

(i) By putting $\tau = 1$ and $e^{-xy} = 1$ in the integral (5) the M-series reduces to the well known generalized hypergeometric function ${}_P F_Q$ and we find the integral [16, p. 63 (eq. 4.4.2)]

(ii) When replacing $r = 1$ the I-function reduces to Fox's H-function and we find the integral [20, p. 61 (eq.5.2.1)] in terms of Fox's H-function.

5. Conclusion:

Our findings are very useful and applicable in mathematical physics and engineering sciences. As a result above outcomes of this paper are easily converted in terms of other special functions after some suitable parametric replacement, such as in Fox's H-function, Maitland's generalized Bessel function and Maitland's generalized hypergeometric function exponential function binomial function and Higher transcendental functions etc.

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Polynomials' Irreducibility Coefficient's whose values are integers

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Abstract

In this article, we provide a method for determining if a given polynomial with int eger coefficients is irreducible over the space of rational numbers. We examine m ore contemporary tests, such as those of Ram Murty, Chen et al., Filaseta, and oth ers, in addition to more established ones like the Eisenstein criterion and irreducib ility over prime finite fields.

Introduction.

If a polynomial can be expressed as a product of lower degree polynomials with coefficients in a certain field, it is said to be reducible over that field. If not, it is considered irreducible.

We will focus on the polynomials with integer coefficients and their irreducibility over the field of rational numbers in this article (which is denoted by Q). The ring of polynomials with integer coefficients is denoted by Z[X].

Determining whether or not a specific polynomial is irreducible is a matter that interests us. Hence, a straightforward test or criterion that would provide this information is preferred. Regrettably, no such test or irreducibility criterion has yet been discovered that will apply to all classes of polynomials; nonetheless, several tests have been discovered that provide useful information for some specific classes of polynomials. Unless otherwise stated, the irreducibility of f(X)will be over Q throughout the article.

According to Eisenstein [1], the most widely used irreducibility criterion is as follows:

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in Z[x]$ if $\exists$ a prime p such that$p \nmid a_\mathrm{n} p/a_0, p/a_1, \dots p/a_{n-1}$ and$p^2 \nmid a_0$, then $f(x)$ is irreducible over Q.

Eisenstein polynomials are such a polynomial, f(X). It frequently occurs that this condition does not directly apply to a given polynomial f(X), but that it might apply to f(X+a) for some constant a. As a result, we experiment with different choices of an in an effort to convert f(X) into a polynomial that meets the criterion's requirements.

It is known that practically all polynomials with integer coefficients are irreducible polynomials according to the probabilistic Galois theory. It seems sense to search for additional criteria to demonstrate the irreducibility of a particular polynomial.

But, there are alternative conditions that are even simpler than Eisenstein's, and these always hold true, assuming we are prepared to factor some huge numbers.

This article's goal is to explain how to use these criteria to determine whether or not a particular polynomial with integer coefficients is irreducible. For earlier findings, we cite [2], [3], [4], [5] and [6].

Preliminary

Let R be an integnal domain. A polynomial $g(x) \in R[x]$ of positive degree [deg(g(x)) ≥ 1] is said to be an irreducible polynomial over R if it can not be expressed as product of two polynomials of positive degree.

A polynomial of positive degree which is not irreducible is called reducible over R.

Example. Consider the polynomial f$(x) = 3x^2 + 3$. Since it can not be expressed as product of two positive degree polynomials in $Z[x]$

We notice it is irreducible polynomial over Z .

Again,

$$3x^2 + 3 = 3(x^2 + 1) = \text{Product of two polynimials}$$
$$= g(x)h(x) \text{ (say)}$$

We find $3x^2 + 3$ can be expressed as product of two non-units and thus $f(x) = 3x^2 + 3$ is not an irreducible element in $Z[x]$.

Methodology

Theorem-1. Every irreducible elements in $R[x]$ is an irreducible polynomal where R is an integral domain with unity.

Proof: Let $f(x) \in R[x]$ be any irreducible elements.

Suppose $f(x)$ is reducible polynomial

Then $f(x) = g(x)h(x); g(x), h(x) \in R[x]$, where degree $g(x) > 0$, degree $h(x) > 0$

⇒ degree $g(x) > 0, degree\ h(x) > 0$ are not constant polynomials:

∴ $g, h \notin R$

⇒ g, h cannot be units in R

⇒ g, h cannot be units in $R[x]$

⇒ $f(x)$ is not irreducible element.

This contradiction proves our result.

Irreducibility Criteria

1. If a polynomial degree$f(x) > 1$ and $f(a) = 0$ for some $a \in F$. Then $f(x)$ is reducible over F, where F is a field
2. Reducibility Test for degree 2 and 3 : Let F be a field if $f(x) \in F[x]$ and deg $f(x) = 2$ or 3 then f(x) is irreducible over F if and only if $f(x)$ has a zero in F.

Example:

(a)$f(x) = 2x^2 + 4 \in \mathbb{R}[x]$ Since $f(x)$ has no zero in $\mathbb{R}$

$\Rightarrow f(x)$ is irreducible over $\mathbb{R}$

But, it is reducible over C.

(b)$f(x) = x^2 - 2 \in Q[x]$ is irreducible over Q, since $f(x)$ has no zero in Q. But it is reducible over R.

(c)$f(x) = x^2 + 1$ is irreducible over Z_3, but reducible over $\mathbb{Z}_5$.

$Z_3 = \{0, 1, 2\}$ and $Z_5 = \{0, 1, 2, 3, 4\}$

$$f(0) = 0^2 + 1 = 1$$
$$f(0) = 1^2 + 1 = 2$$
$$f(2) = 2^2 + 1 = 5$$

$f(x) = x^2 + 1$ is irreducible over Z_3,

$f(3) = 3^2 + 1 = 10(\text{mod}5) = 0$

Hence it is reducible over $\mathbb{Z}_5$.

3. Let $f(x) \in \text{Z}[x]$ if $f(x)$ is reducible over Z, then it is reducible over Q.
4. Mod p irreducibility Test: Let p be a prime and suppose that $f(x) \in \mathbb{Z}[x]$ with $\deg f \geq 1$. Let $f_p(x)$ be the polynomial in $Z_p[x]$ obtained from $f(x)$ by reducing all the co-efficients of $f(x)$ modulo p. if $f(x)$ is irreducible over Z_p and $\deg f(x) = \deg f_p(x)$, then $f(x)$ is irreducible over Q.

For example, consider $f(x) = 21x^3 - 3x^2 + 2x + 8$

Then $f_2(x) = x^3 + x^2 \in z_2[\text{x}]$

$f_2(x) = x^2(x + 1)$ is reducible

But $f(x)$ is irreducible over Q.

this shows that mod p irreducibility test may fail for some p and work for others.

(a) $f(x) = 21x^3 - 3x^2 + 2x + 9$ then over Z_2,

we have $f_2(x) = x^3 + x^2 + 1$

$f_2(0) = 1$ and $f_2(1) = 1$

$\Rightarrow f_2(x)$ is irreducible over Z_2.

$\Rightarrow f(x)$ is irreducible over Q.

(b) $f(x) = 6x^2 + 8x^2 + 6x - 4$ then over Z_5 is

$f_5(x) = x^3 + 3x^2 + x - 4$

$f_5(0) = 1,\ f_5(1) = 1, f_5(2) = 3,\ f_5(3) = 3, f_5(4) = 2$

$\Rightarrow f_5(x)$ irreducible over $Z_5 \Rightarrow f(x)$ is irreducible over Q.

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Quasi Weakly Essential Supplemented Modules: An Overview

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ABSTRACT:

In this present work supplemented modules are extended as quasi weakly essential supplemented modules and some properties are studied. Also proved that every finite sum of quasi weakly essential supplemented modules is quasi weakly essential supplemented.

Keywords: Essential Submodules, Small Submodules, Weakly essential supplemented module.

Mathematics Subject Classification: 16D10, 16D70, 16D99

1. INTRODUCTION:

By using the basic definitions of Alizade R and others [1,2]we are providing sum results:

For any R-module M

- If every essential submodule of M has a weak supplement in M, then M is called a weakly essential supplemented module. Hence obviously every weakly supplemented module is quasi weakly essential supplemented module.
- Every (generalized) hollow and every local module are quasi weakly essential supplemented.
- If $V \leq M$ and if V is a weak supplement of an essential submodule in M, then V is called a quasi weak essential supplement submodule in M.
- If every essential submodule of M is a weak supplement in M, then M is quasi weakly essential supplemented. Therefore it is clear that if every essential submodule of M is a weak essential supplement in M, then M is quasi weakly essential supplemented.
- Let M be a weakly essential supplemented R-module. If every non zero submodule of M is essential in M, then M is quasi weakly supplemented..Also M is weakly supplemented.

- It known that every factor module and every homomorphic image of a weakly essential supplemented module are quasi weakly essential supplemented.
- Let M be an R-module, U⊴ M and K≤ M. If K is weakly essential supplemented and U+K has a weak g-supplement in M, then U has also a quasi weak supplement in M.
  By exploiting the above fact we have proved that the finite sum of weakly essential supplemented modules is quasi weakly essential supplemented.
- Let M be a weakly essential supplemented module. Then M/RadN have no proper essential submodules.
- Let M be a weakly essential supplemented R-module. Then every finitely M-generated R-module is quasi weakly essential supplemented.

Present paper includes **section 1** introduction , **section 2**basic definitions, existing propositions,lemmas and our results using them, **section 3**conclusion.

2. LETRATURE REVIEWED & RESULTS:

Definition 2.1 Let M be an R-module and U, V ⊆ M. V is called weak essential supplement of U if U + V = M and U ∩ V «M.

Definition 2.2 Let M be an R-module. If every essential submodule of M has a weak supplement in M, then M is called quasi weakly essential supplemented module.

Example 2.1 Supplemented, artinian, semisimple, linearly compact, uniserial and hollow modules are quasi weakly essential supplemented modules.

Proposition 2.1 Every factor module of a weakly essential supplemented module is quasi weakly essential supplemented.

Proof: Let M/K be a factor module of a weakly essential supplemented module M and L/K ⊆M/K. Since M is weakly essential supplemented there exists essential submodule N of M such that L + N = M and L ∩ N « M. Then M/K = (L + N)/K = L/K + (N + K)/K and (L/K)∩((N +K)/K)«M/K since L∩K «M.

Proposition 2.2 A small cover of a weakly essential supplemented module is a quasi weakly essential supplemented module.

Proof: Let M be a small cover of a weakly essential supplemented module N. ThenN ≅ M/K for some K«M. Take a submodule L of M and a weak essential supplement X/K of (L+K)/K in M/K. Since K«M, we get (X∩L)+K=X∩(L+K)«M and X is a weak supplement of L in M. Thus M is quasi weakly essential supplemented.

In the following proposition there are some properties of quasi weakly essential supplemented modules.

Proposition 2.3 Let M be an R-module. If M is quasi weakly essential supplemented, then the following properties hold:

(i) M is semi local;
(ii) $M = M_1 \oplus M_2$ with M_1 semisimple and Rad (M) $\trianglelefteq M_2$;
(iii) Every supplement in M and every direct summand of M is quasi weakly essential supplemented.

Proof (i) and (ii) follow from Propositions i.e for a proper sub module N of M, the following are equivalent:
(i) M/N is semi simple;
(ii) For every $L \subseteq M$ there exists a sub module $K \subseteq M$ such that L+K = M and $L \cap K \subseteq N$;
(iii) There exists a decomposition $M = M_1 \oplus M_2$ such that M_1 is semi simple, $N \trianglelefteq M_2$ and M_2/N is semisimple. Since for every $L \subseteq M$ there exists a weak supplement $K \subseteq M$ such that L+K=M and $L \cap K \subseteq R$ ad(M).
(iv) Let $N \subseteq M$ be a supplement of M. Then N+K=M and $N \cap K \ll N$ for some $K \subseteq M$. By Proposition 2.1. ,$M/K \cong N/N \cap K$ is weakly essential supplemented and by Proposition 2.2, N is weakly essential supplemented. Direct summands are essential supplements and so they are quasi weakly essential supplemented.

Lemma 2.1 Let M be an R-module with essential submodules K and M_1. Assume M_1 is quasi weakly essential supplemented and M_1 + K has a quasi weak essential supplement in M. Then K has a quasi weak essential supplement in M .

Proof: Let X be a quasi weak essential supplement of M_1 + K in M, i.e.
$M = M_1 + K + X$ and $(M_1 + K) \cap X \ll M$
and let Y be a quasi weak essential supplement of $(K + X) \cap M_1$ in M_1, i.e.
$M_1 = (K + X) \cap M_1 + Y$ and $((K + X) \cap M_1) \cap Y \ll M_1$.
Since $Y \subseteq M_1$,
$Y + K \subseteq M_1 + K \Rightarrow (Y + K) \cap X \subseteq (M_1 + K) \cap X$.
Thus $(Y + K) \cap X \ll M$ since $(M_1 + K) \cap X \ll M$. Now
$M = M_1 + K + X = ((K + X) \cap M_1) + Y + K + X = Y + K + X$ and
$Y \cap (K + X) = Y \cap M_1 \cap (K + X) \ll M_1 \subseteq M$.
Hence Y is a quasi weak essential supplement of K + X in M. Then we obtain
$(X + Y) \cap K \subseteq (X \cap (Y + K)) + (Y \cap (K + X)) \ll M \Rightarrow (X + Y) \cap K \ll M$.
Therefore X+Y is a quasi weak essential supplement for K in M.

Proposition 2.3 Let $M = M_1 + M_2$, where M_1 and M_2 are weakly essential supplemented, then M is quasi weakly essential supplemented.

Proof: Let U be a essential submodule of M. Then $M = U + M_1 + M_2$. Since 0 (zero) submodule is a weak essential supplement of $U+M_1+M_2$ and M_1 is weakly essential supplemented, $U + M_2$ has a weak essential supplement by Lemma 2.1. Hence U has a weak essential supplement since M_2 is quasi weakly essesntial supplemented again byLemma2.1.

Corollary 2.1 Every finite sum of weakly essential supplemented modules is weakly essential supplemented.

Proposition 2.4 Let M be an R-module. If M is quasi weakly essential supplemented, then every finitely M-generated module is quasi weakly essential supplemented.

Proof: Let N be a finitely M-generated module. Then there exists an epimorphism $\bigoplus_F \xrightarrow{f} N \longrightarrow 0$ such that F is finite. Since M is quasi weakly essential supplemented, a finite sum of M is also quasi weakly essential supplemented. By first isomorphism theorem $\bigoplus_F M/Kerf \cong N$. Since every factor module of a weakly essential supplemented module is quasi weakly essential supplemented $\bigoplus_F M/Kerf$ is weakly essential supplemented. Hence N is quasi weakly essential supplemented.

Theorem 2.1 Let $0 \to L \to M \to N \to 0$ be a short exact sequence for R-modules L, M, N. If L and N are weakly essential supplemented and L has a weak essential supplement in M, then M is quasi weakly essential supplemented.

If L is co-closed, then the converse holds; that is if M is weakly essential supplemented, then L and N are quasi weakly essential supplemented.

Proof: Without loss of generality we will assume $L \subseteq M$. Let S be a weak essential supplement of L in M, i.e. $L + S = M$ and $L \cap S \ll M$. Then we have,

$M/L \cap S \cong L/L \cap S \oplus S/L \cap S$.

$L/L \cap S$ is weakly essential supplemented as a factor module of L which is weakly essential supplemented. On the otherhand

$S/L \cap S \cong M/L \cong N$

is weakly essential supplemented. Then $M/L \cap S$ is weakly essential supplemented module as a sum of weakly supplemented modules. Therefore M is quasi weakly essential supplemented by Proposition2.2.

Conversely, if L is co-closed, for $L \cap S \subseteq L$, $L \cap S \ll M$ implies $L \cap S \ll L$ i.e. L is a supplement of S in M. Then by Proposition 2.3 (iv), L is quasi weakly essential supplemented and by Proposition 2.1, N is quasi weakly essential supplemented.

Proposition2.5 Let M be an R-module. M is quasi weakly essential supplemented if and only if

$M / \left(\bigoplus_{i=1}^{n} \right) L_i$ where each L_i is a hollow submodule of M.

Proof:
($\Leftarrow$) part is clearly holds and $\Leftarrow$ and to prove let us suppose n = 1 and M/L is weakly essential supplemented. Consider the following exact sequence: 0 → L → M → M/L → 0.
Case1: If L« M, then M is quasi weakly essential supplemented since it is small cover of M/L
Case 2: If L $\not\ll$ M, then M = L + T for a proper submodule T of M. Since L is hollow L∩T«L⊆M. Hence T is a weak essential supplement of L in M. Since M/L and L are weakly essential supplemented, by Theorem 2.1 M is quasi weakly essential supplemented.

Now suppose it holds when i< n. Let $M/\left(\bigoplus_{i=1}^{n}\right)L_i$ be weakly essential supplemented. We get the following exact sequence:

$$0\rightarrow\left(\bigoplus_{i=1}^{n}L_i\right)/\left(\bigoplus_{i=1}^{n-1}L_i\right)\rightarrow M/\left(\bigoplus_{i=1}^{n-1}L_i\right)\rightarrow M/\left(\bigoplus_{i=1}^{n}L_i\right)\rightarrow 0$$

Since $\left(\bigoplus_{i=1}^{n}L_i\right)/\left(\bigoplus_{i=1}^{n-1}L_i\right)\cong L_n$, is a ^{i} hollow submodule of $M/\left(\bigoplus_{i=1}^{n-1}L_i\right)$ and $M/\left(\bigoplus_{i=1}^{n}L_i\right)$ is essential weakly supplemented, $M/\left(\bigoplus_{i=1}^{n-1}L_i\right)$ is weakly essential supplemented. Therefore, M is quasi weakly essential supplemented by induction.

Corollary 2.2 Let M be an R-module. M is quasi weakly essential supplemented if and only if $M/\left(\bigoplus_{i=1}^{n}L_i\right)$ is quasi weakly essential supplemented where each L_i is a local submodule of M.
Proof: Since local modules are hollow, the proof is obliviously by Proposition2.5.

Corollary 2.3 Let M be an R-module. If Soc(M) is finitely generated, then M/ Soc(M) is quasi weakly essential supplemented if and only if M is quasi weakly essential supplemented.
Proof: Since simple modules are local, the proof is automatically by Corollary 2.2.

Corollary 2.4 Let M be an R-module. M is quasi weakly essential supplemented iff M/S is quasi weakly essential supplemented for a finitely generated supplemented essential submodule S of M.

Proof: Since finitely generated supplemented modules are the irredundant sum of local submodules, the proof is clear by Corollary 2.2.

Lemma 2.2 Let M be a finitely generated module with zero radical and let N be a non-finitely generated essential submodule of M. Then N does not have any quasi weak essential supplement in M.

Proof: Suppose that L is a weak essential supplement of N in M, i.e. $M = N + L$ and $N \cap L \ll M$. Now $N \cap L \subseteq Rad(M) = 0$. Hence $M = N \oplus L$ and N is finitely generated, a contradiction.

Definition2.3 An R-module is called decomposable if it is a direct sum of cyclic modules and finitely generated torsion-free modules of rank one. If R is a principal ideal domain, then a decomposable module is exactly a direct sum of cyclic modules.

Definition 2.4 Let M be an R-module. A submodule N is called pure if $rN = N \cap rM$ for every $r \in R$.

Theorem2.2 Let R be a Dedekind domain,M be an R-module and S be a pure submodule such that M/S is decomposable. Then S is a direct summand of M.

As a result of this theorem, the following corollary can be given.

Corollary 2.5 Let R be a Dedekind domain, L, M, N be R-modules and

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

Be an exact sequence with L pure in M and N decomposable .L and N are quasi weakly essential supplemented if and only if M is quasi weakly essential supplemented.

Proof: ($\Rightarrow$) By Theorem 2.2, the sequence is splitting so M is quasi weakly essential supplemented since L and N are quasi weakly essential supplemented.

($\Leftarrow$) Since direct summands of quasi weakly essential supplemented modules are quasi weakly essential supplemented, L and N are quasi weakly essential supplemented.

3. CONCLUSION: Here we have defined more definitions and example and results related to supplemented module as weakly essential supplemented module.

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FIXED POINTS OF NON-NEWTONIAN EXPANSIVE MAPPINGS

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ABSTRACT

In this study, a novel class of non-Newtonian expansive mappings on non-Newtonian metric spaces is introduced, and various fixed-point theorems are demonstrated for two of these mappings on non-Newtonian metric spaces. Our findings expand upon and generalize a number of earlier findings in the literature.

KEYWORDS: non-Newtonian metric space; non-Newtonian expansive mapping; fixed point.

1. INTRODUCTION

Grossman and Katz were the first to introduce the concept of non-Newtonian calculus [1]. Later, Bashirov et al. [2], Ozyapici et al. [3], Cakmak and Basar [4], and others [5-17] study the non-Newtonian calculus. Non-Newtonian metric has been examined by Cakmak and Basar [4]. They are supported by [7] in a number of ways. The contractive mapping was defined in non-Newtonian metric space by Binbasoglu et al. [18]. A very fascinating area of inquiry in fixed point theory is the study of expansive maps. Wang et al. [19] defended several fixed-point findings in entire metric spaces and deputised the idea of extending maps.For two expansive mappings, Daffer and Kaneko [20] attested to some common fixed-point findings in whole metric spaces. We direct the reader to [21-26] for additional information.

This article presents several fixed-point results in non-Newtonian metric space and introduces the idea of non-Newtonian expansive mappings. Additionally, some earlier results are generalized by these results.

2. PRELIMINARIES

A generator is an injective function that has $\mathbb{R}$, the set of all real numbers, as its domain and a subset of $\mathbb{R}$ as its range. Every generator produces exactly one form of arithmetic, and vice versa, every generator produces a particular type of arithmetic. As a generator, we choose the function exp from $\mathbb{R}$ to the set $\mathbb{R}^+$ of positive reals, that is to say,

$$\beta: \mathbb{R} \longrightarrow \mathbb{R}^+,$$
$$r \longmapsto \beta(r) = e^r = s$$
and
$$\beta^{-1}: \mathbb{R}^+ \longrightarrow \mathbb{R},$$
$$s \longmapsto \beta^{-1}(s) = \ln s = r$$

If $I(r) = r$for all $r \in \mathbb{R}$, then I is called identity function and we know that inverse of the identity function is itself. If $\beta = I$, thenβ generates the classical arithmetic and if $\beta = exp$, then β generates geometrical arithmetic. All concepts of β-arithmetic have similar properties in classical arithmetic.β-zero,β-one and all β-integers are formed as

$$\ldots\ldots, \beta(-2), \beta(-1), \beta(0), \beta(1), \beta(2)\ldots\ldots..$$

The β-positive numbers are the numbers $q \in A$ such that $\dot{0} \dot{<} q$and the β-negative numbers are those for which $q \dot{<} \dot{0}$. The β-zero, $\dot{0}$, and the β-one, $\dot{1}$, turn out to be $\beta(0)$ and $\beta(1)$. The β-integers consist of $\dot{0}$ and all the numbers that result by successive β-addition of $\dot{1}$ and $\dot{0}$ and by successive β-subtraction of $\dot{1}$ and $\dot{0}$.
We denote by $\mathbb{R}(N)$ the range of generator β and write $\mathbb{R}(N) = \{\beta(r) : r \in \mathbb{R}\}$. $\mathbb{R}(N)$ is called Non-Newtonian real line.Non-Newtonian arithmetic operations on $\mathbb{R}(N)$ are represented as follows:

β-addition $\quad p \dot{+} q = \beta\left(\beta^{-1}(p) + \beta^{-1}(q)\right)$,
β-subtraction $\quad p \dot{-} q = \beta\left(\beta^{-1}(p) - \beta^{-1}(q)\right)$,
β-multiplication $\quad p \dot{\times} q = \beta\left(\beta^{-1}(p) \times \beta^{-1}(q)\right)$,
β-division $\quad p \dot{/} q = \beta(\beta^{-1}(p)/\beta^{-1}(q))$,
β-order $\quad p \dot{<} q (p \dot{\leq} q) \Leftrightarrow \beta^{-1}(p) < \beta^{-1}(q)\left(\beta^{-1}(p) \leq \beta^{-1}(q)\right)$,

The β-square of a number $p \in A \subset \mathbb{R}(N)$ is denoted by $p \dot{\times} p = p^{2_N}$. For each β-nonnegative number v, the symbol $\sqrt{p}^N$ will be used to denote $v = \beta\left(\sqrt{\beta^{-1}(p)}\right)$ which is the unique β-square is equal to p, which means that $v^{2_N} = p$. Throughout this paper, p^{p_N}denotes the pth non-Newtonian exponent. Thus we have

$$p^{2_N} = p \dot{\times} p = \beta\left(\beta^{-1}(p) \times \beta^{-1}(p)\right) = \beta([\beta^{-1}(p)]^2),$$
$$p^{3_N} = p^{2_N} \dot{\times} p = \beta\left(\beta^{-1}(p^{2_N}) \times \beta^{-1}(p)\right)$$
$$= \beta\left(\beta^{-1}\left(\beta\left(\beta^{-1}(p) \times \beta^{-1}(p)\right)\right) \times \beta^{-1}(p)\right) = \beta([\beta^{-1}(p)]^3),$$
$$\ldots.$$
$$p^{p_N} = p^{p-1_N} \dot{\times} p = \beta([\beta^{-1}(p)]^p)$$
$$\vdots$$

The β-absolute value of a number $p \in A \subset \mathbb{R}(N)$ is defined as $\beta(|\beta^{-1}(p)|)$ and is denoted by $|p|_N$. For each number $p \in A \subset \mathbb{R}(N), \sqrt{p^{2_N}}^N = |p|_N = \beta(|\beta^{-1}(p)|)$.
In this case,

$$|p|_N = \begin{cases} p, & \text{if} p \dot{>} \dot{0} \\ \dot{0}, & \text{if} p = \dot{0} \\ \dot{0} \dot{-} p, & \text{if} p \dot{<} \dot{0} \end{cases}$$

Also $\mathbb{R}^{+}(N)$ denotes non-Newtonian positive real numbers and $\mathbb{R}^{-}(N)$ denotes non-Newtonian negative real numbers. β-intervals are represented by

Closed β-interval $\dot{[}p,q\dot{]} = [p,q]_N = \{s \in \mathbb{R}(N) : \ p \dot{\leq} s \dot{\leq} q\}$
$= \{s \in \mathbb{R}(N) : \ \beta^{-1}(p) \leq \beta^{-1}(s) \leq \beta^{-1}(q)\}$

Open β-interval $\dot{(}p,q\dot{)} = (p,q)_N = \{s \in \mathbb{R}(N) : \ p \dot{<} s \dot{<} q\}$
$= \{s \in \mathbb{R}(N) : \ \beta^{-1}(p) < \beta^{-1}(s) < \beta^{-1}(q)\}$

Likewise, semi-closed and semi-open β-intervals can be represented. For the set $\mathbb{R}(N)$ of non-Newtonian real numbers, the binary operations $(\dot{+})$addition and $(\dot{\times})$ multiplication are defined by

$$\dot{+} \ : \ \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(p,q) \longmapsto p \dot{+} q = \beta\big(\beta^{-1}(p) + \beta^{-1}(q)\big)$$
$$\dot{\times} \ : \ \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(p,q) \longmapsto p \dot{\times} q = \beta\big(\beta^{-1}(p) \times \beta^{-1}(q)\big).$$

The fundamental properties provided in the classical calculus is provided in non-Newtonian calculus, too.

Lemma 2.1 (see [4]). $(\mathbb{R}(N),\dot{+},\dot{\times})$ is a topologically complete field.

Lemma 2.2 (see [4])$|p \dot{\times} q|_N = |p|_N \dot{\times} |q|_N \forall \, p,q \in \mathbb{R}(N)$.

Lemma 2.3 (see [4]) $|p\dot{+}q|_N \dot{\leq} |p|_N \dot{+} |q|_N, \forall \, p,q \in \mathbb{R}(N)$

The non-Newtonian metric spaces provide an alternative to the metricspaces introduced in [4].

Definition 2.4 (see [4]). Let X be a non-empty set and $d_N : X \times X \longrightarrow \mathbb{R}^{+}(N)$be a function such that for all $p,q,k \in X$;

(NNM1). $d_N(p,q) = \dot{0} \Leftrightarrow p = q$

(NNM2). $d_N(p,q) = d_N(q,k)$

(NNM3). $d_N(p,q) \dot{\leq} d_N(p,k) \dot{+} d_N(k,q)$.

Then, the map d_N is called non-Newtonian metric and the pair $(X, d_N)$is called non-Newtonian metric space.

Definition 2.5 (see [4]). Let Xbe a vector space on $\mathbb{R}(N)$. If a function $\|\,.\,\|_N$: $X \longrightarrow \mathbb{R}^{+}(N)$ satisfies the following axioms for all $p,q \in X$ and $\lambda \in \mathbb{R}(N)$:

(NNN1). $\|p\|_N = \dot{0} \Leftrightarrow p = \dot{0}$

(NNN2). $\|\lambda \dot{\times} p\|_N = |\lambda|_N \dot{\times} \|p\|_N$

(NNN3). $\|p \dot{+} q\|_N \dot{\leq} \|p\|_N \dot{+} \|q\|_N$.

then it is called a non-Newtonian norm on $X$and the pair $(X, \|\,.\,\|_N)$ is called a non-Newtonian normed space.

Remark 2.6 (see [4]). Here it is easily seen that every non-Newtonian norm $\|\,.\,\|_N$on Xproduces a non-Newtonian metric $d_N$on Xgiven by

$$d_N(p,q) = \|p \dot{-} q\|_N, \forall \, p,q \in X$$

Definition 2.7 (see [4]). (non-Newtonian convergent sequence) A sequence $\{q_n\}$ in a non-Newtonian metric space (X, d_N) is said to be non-Newtonian convergent if for every given $\epsilon \dot{>} \dot{0}$, there exists an $n_0 = n_0(\epsilon) \in \mathbb{N}$ and $q \in X$ such that $d_N(q_n, q) \dot{<} \epsilon$ for all $n > n_0$ and is denoted by ${}^{\mathrm{N}}\lim_{n\to+\infty} q_n = q$ or $q_n \xrightarrow{N} q$ as $n \longrightarrow \infty$.

Definition 2.8 (see [4]). (non-Newtonian Cauchy sequence) A sequence $\{q_n\}$ in a non-Newtonian metric space (X, d_N) is said to be non-Newtonian Cauchy if for every given $\epsilon \dot{>} \dot{0}$, there exists an $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $d_N(q_n, q_m) \dot{<} \epsilon$ for all $m, n > n_0$.

Definition 2.9 (see [4]). (non-Newtonian complete metric space) The space X is said to be non-Newtonian complete if every non-Newtonian Cauchy sequence in X converges.

Definition 2.10 (see [4]). (non-Newtonian bounded) Let (X, d_N) be a non-Newtonian metric space. The space X is said to benon-Newtonian bounded if there is a non-Newtonian constant $\kappa \dot{>} \dot{0}$ such that $d_N(p, q) \dot{\leq} \kappa$ for all $p, q \in X$. The space X is said to be non-Newtonian unbounded if it is not non-Newtonian bounded.

Proposition 2.11 (see [4]). Suppose that the non-Newtonian metric d_N on$\mathbb{R}(N)$is such that $d_N(p, q) = |p \dot{-} q|_N$ for all $p, q \in \mathbb{R}(N)$, then $(\mathbb{R}(N), d_N)$ is a non-Newtonian metric space.

Lemma 2.12 (see [18]). Let (X, d_N) be a non-Newtonian metric space. Then,

(1). A non-Newtonian convergent sequence in X is non-Newtonian bounded and its non-Newtonian limit is unique.

(2). A non-Newtonian convergent sequence in X is a non-Newtonian Cauchy sequence in X.

From the definition of non-Newtonian Cauchy sequence and Lemma 2.12, we can give the following corollary:

Corollary 2.13 (see [18]) A non-Newtonian Cauchy sequence is non-Newtonian bounded.

Lemma 2.14 (see [18]) Suppose (X, d_N) is a non-Newtonian metric space and $p, q, k \in X$. Then

$$|d_N(p.q) \dot{-} d_N(q, k)|_N \dot{\leq} d_N(p, k)$$

Definition 2.15 Let Xbe a set and Υa map from $X$to X. A fixed point of $\Upsilon$is asolution of the functional equation$\Upsilon(q) = q, q \in X$. A point $q \in X$ is called common fixed point of two self-mappings Υ and g on X if $\Upsilon(q) = g(q) = q$.

Definition 2.16 (see [18]) Suppose(X, d_N) is a non-Newtonian complete metric space. A mapping $\Upsilon: X \to X$is called non-Newtonian Lipschitzian if there exists a non-Newtonian number $\delta \in \mathbb{R}(N)$ such that

$d_N(\Upsilon(p), \Upsilon(q)) \dot{\leq} \delta \dot{\times} d_N(p, q), \forall\, p, q \in X.$

The mapping $\Upsilon$is called non-Newtonian contractive if $\delta \dot{<} \dot{1}$.

Binbasıoglu et al [18] established following result in non-Newtonian metric space.
Theorem 2.17 Let Υ be a non-Newtonian contraction mapping on a non-Newtonian complete metric space X. Then Υ has a unique fixed point.
Proposition 2.18(see[27])The non-Newtonian distance is commutative.
Proposition 2.19(see[27])Let(X, d_N) be a non-Newtonian metric space and let $p, q, k, l \in X$. Then

$$|d_N(p,q) \dot{-} d_N(k,l)|_N \dot{\leq} d_N(p,k) \dot{+} d_N(q,l)$$

Proposition 2.20(see[27])Let$(X, \|\,.\,\|_N)$ be a non-Newtonian normed space. Then

$$|\|p\|_N \dot{-} \|q\|_N|_N \dot{\leq} \|p \dot{-} q\|_N, \forall p, q \in X$$

Definition 2.21(see[27])Suppose (X, d_N) is a non-Newtonian complete metric space. A mapping $\Upsilon: X \to X$is called non-Newtonianexpansive if there exists a non-Newtoniannumber $\delta \dot{>} \dot{1}$ such that
$d_N(\Upsilon\eta, \Upsilon\xi) \dot{\geq} \delta \dot{\times} d_N(\eta, \xi), \forall\, \eta, \xi \in X.$
Lemma 2.22(see[27]) Let $\{q_n\}$ be a sequence in a non-Newtonian metric space such that $d_N(q_n, q_{n+1}) \leq \delta \dot{\times} d_N(q_{n-1}, q_n)$, where $\delta \dot{<} \dot{1}$ and $n \in \mathbb{N}$.Then $\{q_n\}$ is a non-Newtonian Cauchy sequence in X.

3. MAIN RESULTS:

Now, we give some fixed-point results for non-Newtonian expansive mappings in a non-Newtonian complete metric space. Our first main result as follows.
Theorem 3.1 Let (X, d_N) be a non-Newtonian complete metric space and Υ a continuous mapping satisfying the following condition:
$d_N(\Upsilon\eta, \Upsilon\xi) \dot{\geq} \mu \dot{\times} (d_N(\eta, \Upsilon\eta) \dot{\times} [1 \dot{+} d_N(\xi, \Upsilon\xi)]) \dot{/} (1 \dot{+} d_N(\eta, \xi)) \dot{+} \lambda \dot{\times} d_N(\eta, \xi)$ (2.1)
for all $\eta, \xi \in X, \eta \neq \xi$, where $\mu, \Lambda \dot{\geq} 0$ are constants and$\Delta \dot{+} \Lambda \dot{>} 1, \Lambda \dot{>} 1$. Then, Υ has a fixed point in X.
Proof Choose$\eta_0 \in X$ be arbitrary, to define the iterative sequence $\{\eta_n\}_{n \in \mathbb{N}}$ as follows and $\Upsilon\eta_n = \eta_{n-1}$ for $n = 1,2,3, ...$ Then, using (2.1), we obtain

$$d_N(\Upsilon\eta_{n+1}, \Upsilon\eta_{n+2}) \dot{\geq} \Delta \dot{\times} (d_N(\eta_{n+1}, \Upsilon\eta_{n+1}) \dot{\times} [1 \dot{+} d_N(\eta_{n+2}, \Upsilon\eta_{n+2})]) \dot{/} (1 \dot{+} d_N(\eta_{n+1}, \eta_{n+2})) \dot{+} \Lambda \dot{\times} d_N(\eta_{n+1}, \eta_{n+2})$$

$$\Rightarrow \quad d_N(\eta_n, \eta_{n+1}) \dot{\geq} \Delta \dot{\times} (d_N(\eta_{n+1}, \eta_n) \dot{\times} [1 \dot{+} d_N(\eta_{n+2}, \eta_{n+1})]) \dot{/} (1 \dot{+} d_N(\eta_{n+1}, \eta_{n+2})) \dot{+} \Lambda \dot{\times} d_N(\eta_{n+1}, \eta_{n+2})$$

$$\Rightarrow \quad d_N(\eta_n, \eta_{n+1}) \dot{\geq} \Delta \dot{\times} d(\eta_n, \eta_{n+1}) \dot{+} \Lambda \dot{\times} d_N(\eta_{n+1}, \eta_{n+2})$$

$$\Rightarrow d_N(\eta_n, \eta_{n+1}) \dot{\geq} \Delta \dot{\times} d_N(\eta_{n+1}, \eta_n) \dot{+} \Lambda \dot{\times} d_N(\eta_{n+1}, \eta_{n+2})$$

$$\Rightarrow \quad (1 \dot{-} \Delta) \dot{\times} d_N(\eta_{n+1}, \eta_n) \dot{\geq} \Lambda \dot{\times} d_N(\eta_{n+1}, \eta_{n+2}).$$

The last inequality gives

$$d_N(\eta_{n+1}, \eta_{n+2}) \dot{\leq} (1 \dot{-} \Delta) \dot{/} \Lambda \dot{\times} d_N(\eta_n, \eta_{n+1}) = \delta \dot{\times} d_N(\eta_n, \eta_{n+1}) \qquad (3.2)$$

where $\delta = (1 \dot{-} \Delta) \dot{/} \Lambda$, then we get $\delta \dot{<} \dot{1}$, since$\Delta \dot{+} \Lambda \dot{>} 1$. Repeating this process in condition (3.2), we find

$$d_N(\eta_{n+1}, \eta_{n+2}) \leq \delta^{n+1_N} \dot{\times} d_N(\eta_0, \eta_1) \qquad (3.3)$$

and by Lemma 2.22, $\{\eta_n\}$ is an NN-Cauchy sequence. Since (X, d_N) is non-Newtonian complete,there exists a point q in X such that $\eta_n \xrightarrow{N} \eta$. By continuity of Υ we have,

$$\Upsilon\eta^* = \Upsilon\left(\lim_{n\to\infty}^N \eta_n\right)$$
$$= \lim_{n\to\infty}^N \Upsilon\eta_n$$
$$= \lim_{n\to\infty}^N \eta_{n-1} = \eta^* (3.4)$$

that is, $\Upsilon\eta^* = \eta^*$; thus, Υ has a fixed point in X.

For uniqueness, let ξ^* be another fixed point of Υ in X, then $\Upsilon\xi^* = \xi^*$ and $\Upsilon\eta^* = \eta^*$. Now,

$$d_N(\Upsilon\eta^*, \Upsilon\xi^*)$$
$$\geq \Delta \dot{\times} \left(d_N(\eta^*, \Upsilon\eta^*) \dot{\times} \left[1 \dot{+} d_N(\xi^*, \Upsilon\xi^*)\right]\right) \dot{/} [1 \dot{+} d_N(\eta^*, \xi^*)] \dot{+} \Lambda \dot{\times} d_N(\eta^*, \xi^*)$$

This implies that

$$d_N(\eta^*, \xi^*) \dot{\geq} \Lambda \dot{\times} d_N(\eta^*, \xi^*)$$

That is

$$d_N(\eta^*, \xi^*) \dot{\geq} \Lambda \dot{\times} d(\eta^*, \xi^*)$$
$$\Rightarrow d_N(\eta^*, \xi^*) \dot{\leq} 1 \dot{/} \Lambda \dot{\times} d_N(\eta^*, \xi^*) \qquad (3.6)$$

This is true only when $d_N(\eta^*, \xi^*) = 1$ and so $\eta^* = \xi^*$. Hence Υ has a unique fixed point in X.

Next we prove Theorem 3.1 for surjective mapping.

Theorem 3.2 Let (X, d_N) be a non-Newtonian complete metric space and Υ a surjective mapping satisfying the condition (3.1) for all $\eta, \xi \in X, \eta \neq \xi, \Delta \dot{+} \Lambda \dot{>} 1, \Lambda \dot{>} 1$.Then, Υ has a fixed point in X.

Proof Choose $\eta_0 \in X$ to be arbitrary, and define the iterative sequence $\{\eta_n\}_{n\in\mathbb{N}}$ as follows: $\Upsilon\eta_n = \eta_{n-1}$ for $n = 1,2,3, \ldots.$ Then, using (3.1), we obtain, sequence $\{\eta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. But X is a complete; hence $\{\eta_n\}_{n\in\mathbb{N}}$ is non-Newtonian converges and $\exists$ an element $\eta^* \in X$. Such that $\eta_n \xrightarrow{N} \eta^*$ as $n \to \infty$.

Since Υ is a Surjective map, so there exists a point ξ in X, such that $\eta = \Upsilon\xi$. Consider

$$d_N(\eta_n, \eta) = d_N(\Upsilon\eta_{n+1}, \Upsilon\xi)$$
$$\dot{\geq} \Delta \dot{\times} d_N(\eta_{n+1}, \Upsilon\eta_{n+1}) \dot{\times} [1 \dot{+} d_N(\xi, \Upsilon\xi)] \dot{/} \left(1 \dot{+} d_N(\eta_{n+1}, \xi)\right)$$
$$\dot{+} \Lambda \dot{\times} d_N(\eta_{n+1}, \xi) \qquad (3.5)$$

Taking $n \to \infty$, we get

$$d_N(\eta, \eta) \dot{\geq} \Delta \dot{\times} d_N(\eta, \eta) \dot{\times} [1 \dot{+} d_N(\xi, \eta)] \dot{/} \left(1 \dot{+} d_N(\eta, \xi)\right) \dot{+} \Lambda \dot{\times} d_N(\eta, \xi)$$
$$\Rightarrow \quad 1 \dot{\geq} \Lambda \dot{\times} d_N(\eta, \xi)$$
$$\Rightarrow \quad \Lambda \dot{\times} d_N(\eta, \xi) \dot{\leq} 1$$
$$\Rightarrow \quad d_N(\eta, \xi) = 1 \text{ as } \Lambda \dot{>} 1.$$

Hence $\eta = \xi$ and so $\Upsilon\eta = \eta$, that is, η is a fixed point of Υ.

Now, we prove uniqueness. Let ξ^* be another fixed point of T in X, then $\Upsilon\xi^* = \xi^*$ and $\Upsilon\eta^* = \eta^*$. Now,

$$d_N(\Upsilon\eta^*, \Upsilon\xi^*)$$
$$\geq \Delta \dot{\times} \left(d_N(\eta^*, \Upsilon\eta^*) \dot{\times} \left[1 \dot{+} d_N(\xi^*, \Upsilon\xi^*)\right]\right) \dot{/} [1 \dot{+} d_N(\eta^*, \xi^*)] \dot{+} \Lambda \dot{\times} d_N(\eta^*, \xi^*)$$

This implies that

$$d_N(\eta^*, \xi^*) \dot{\geq} \Lambda \dot{\times} d_N(\eta^*, \xi^*)$$

That is

$$d_N(\eta^*, \xi^*) \dot{\geq} \Lambda \dot{\times} d(\eta^*, \xi^*)$$
$$\Rightarrow d_N(\eta^*, \xi^*) \dot{\leq} 1 \dot{/} \Lambda \dot{\times} d_N(\eta^*, \xi^*)$$

This is true only when $d_N(\eta^*, \xi^*) = 1$ and so $\eta^* = \xi^*$. Hence Υ has a unique fixed point in X. The proof is completed.

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A COMPREHENSIVE OVERVIEW OF RIEMANN INTEGRATION

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Abstract:

This research article discusses the concept of Riemann integration, which is an important topic in calculus. The article begins with an introduction to the historical background of Riemann integration, followed by a detailed explanation of the basic concepts and definitions involved. The article then goes on to discuss the properties of Riemann integrable functions, and finally provides some examples of applications of Riemann integration in various fields. The article is intended to provide a comprehensive understanding of Riemann integration and its significance in calculus.

Keywords: Riemann integration, calculus, integrable functions, properties, applications.

Introduction:

The concept of integration is an important topic in calculus, and Riemann integration is one of the most widely used methods for calculating integrals. Riemann integration was first introduced by Bernhard Riemann in the 19th century, and since then, it has been an important tool for mathematicians and scientists. Riemann integration is a method of approximating the area under a curve using rectangles. The area of each rectangle is calculated by multiplying the width of the rectangle by the height of the curve at a particular point. By adding up the areas of all the rectangles, an approximate value of the area under the curve can be obtained. Riemann integration is based on the concept of partitions. A partition is a finite set of points that divides an interval into subintervals. The Riemann integral of a function f over an interval [a, b] is defined as the limit of a sum of rectangles as the width of the rectangles approaches zero. The limit is taken over all possible partitions of the interval [a, b]. A function is said to be Riemann integrable if the Riemann integral exists and is finite.

One of the key properties of Riemann integrable functions is that they are continuous almost everywhere. This means that the function may have some points where it is discontinuous, but these points have measure zero. Riemann integrable functions also satisfy the fundamental theorem of calculus, which states that the integral of the derivative of a function is equal to the difference between the values of the function at the endpoints of the interval.

1. **Definition of the Riemann integral:** The Riemann integral is defined as the limit of a sum of rectangles that approximates the area under a curve. The integral is denoted by ∫f(x)dx, where f(x) is a function and dx represents an infinitesimal interval in the x-axis.
2. **Integrable functions:** A function is said to be Riemann integrable if it satisfies certain conditions, such as boundedness and continuity almost everywhere.
3. **The Riemann integral in higher dimensions:**We explore the concept of Riemann integration in higher dimensions and discuss the challenges in defining the integral in multiple variables.
4. **Applications of Riemann integration:**

   4.1 Physics: The Riemann integration is used in physics, particularly in calculating the work done by a variable force.

   4.2 Economics: The Riemann integration is used in economics, specifically in calculating consumer surplus.

   4.3 Probability theory: Riemann integration is used in probability theory, particularly in calculating the expected value of a continuous random variable.
5. **Limitations and Challenges of Riemann Integration :** This theory incudes these limitations and challenges:

   5.1 Non-integrable Functions: There are functions that are not Riemann integrable, such as the Dirichlet function. These functions are discontinuous almost everywhere and do not satisfy the conditions required for Riemann integration. In such cases, other integration methods such as Lebesgue integration may be more appropriate.

   5.2 Limited Scope: Riemann integration is limited to functions that are defined on a bounded interval. If a function is defined on an unbounded interval, such as f(x) = 1/x, then the integral may not exist. In such cases, improper integrals can be used to calculate the area under the curve.

   5.3 Limitations in Higher Dimensions: The concept of Riemann integration can be extended to higher dimensions, but the method becomes more complicated due to the increased number of variables involved. The volume of a solid in three dimensions, for example, can be calculated using a triple integral, which involves the integration of a function over a three-dimensional region.

   5.4 Computational Challenges: In practice, it can be challenging to calculate the Riemann integral of a function, especially for complex functions or functions with a large number of variables. Numerical integration methods, such as the trapezoidal rule and Simpson's rule, can be used as an alternative to Riemann integration in such cases.

   5.5 Precision Limitations: Riemann integration involves approximating the area under a curve using rectangles. The accuracy of this approximation depends on the size of the rectangles used. If the rectangles are too large, the approximation may not be accurate enough. If the rectangles are too small, the calculation may take longer and may be subject to numerical errors.

6. **Differences between Riemann integration and other integration methods:**
   - **6.1 Lebesgue integration:**Lebesgue integration is more general than Riemann integration.This method uses a different approach to defining the integral of a function, based on the concept of measure theory. Instead of approximating the area under the curve using rectangles, Lebesgue integration defines the integral of a function in terms of its "measurable" sets, which can include sets that are not necessarily intervals.
   - **6.2 Trapezoidal Rule:** This method approximates the area under the curve of a function by dividing the interval of integration into subintervals and approximating the area within each subinterval by a trapezoid whose two parallel sides are the function values at the endpoints of the subinterval. The sum of the areas of all the trapezoids gives an approximation of the total area under the curve.
   - **6.3 Simpson's Rule:** This method approximates the area under the curve of a function by dividing the interval of integration into subintervals and approximating the area within each subinterval by a parabolic curve that passes through the endpoints and the midpoint of the subinterval. The sum of the areas of all the parabolic curves gives an approximation of the total area under the curve.
   - **6.4 Monte Carlo Integration:** This method approximates the area under the curve of a function by randomly sampling points within the interval of integration and calculating the fraction of the points that lie under the curve. The product of this fraction and the area of the interval of integration gives an approximation of the total area under the curve.

Conclusions: Riemann integration is an important topic in calculus that has numerous applications in various fields. A comprehensive understanding of Riemann integration and its properties is essential for anyone interested in pursuing advanced mathematics. Despite its limitations and challenges, Riemann integration remains a powerful tool for solving problems in mathematics and science, and continues to be an active area of research.

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FIXED POINT RESULTS IN ORDERED S-METRIC SPACES FOR RATIONAL TYPE EXPRESSIONS

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Abstract:

The aim of this paper is to present some fixed point theorems for g-monotone maps involving rational expression in the framework of S-metric spaces endowed with a partial order using a class of pairs of functions satisfying certain assumptions.

Keywords: fixed point; S-metric space; contractions; partially ordered set, altering distance function.

1. Introduction and Preliminaries:

Metric spaces are very important in mathematics and applied sciences. So, some authors have tried to give generalizations of metric spaces in several ways. For example, Gahler [1] and Dhage [2] introduced the concepts of 2-metric spaces and *D*-metric spaces, respectively. In 2006, Mustafa and Sims [3] introduced a new structure of generalized metric spaces which are called *G*-metric spaces as a generalization of metric spaces (X, d) to develop and introduce a new fixed point theory for various mappings in this new structure. Sedghi et al. [4] introduced the notion of a D^*-metric space. Das and Gupta [5] proved the following fixed point theorem.

Theorem 1.1 (see [5]) Let (X, d) be a complete metric space and $f: X \to X$ a mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying

$$d(fx, fy) \leq \alpha \frac{d(y, fy)[1 + d(x, fx)]}{1 + d(x,y)} + \beta d(x, y) \tag{1.1}$$

for all $x, y \in X$. Then f has a unique fixed point in X.

For more details on fixed point results with rational expressions, see [6-8]. Cabrera et al. [9] proved Theorem 1.1 in the context of partially ordered metric spaces.

Definition 1.2 (see [9]) Let $(X, \preceq)$ is a partially ordered set and $f : X \to X$ is said to be monotone non-decreasing if for all $x, y \in X$,

$$x \preceq y \Rightarrow fx \preceq fy. \tag{1.2}$$

Theorem 1.3 (see [9]) Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist a metric d on X such that (X, d) be a complete metric space. Let $f: X \to X$ be a continuous and non-decreasing mapping such that (1.1) is satisfied for all $x, y \in X$ with $x \preceq y$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Theorem 1.4 (see [9]) Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist a metric d on X such that (X, d) be a complete metric space. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$, for all $n \in \mathbb{N}$. Let $f: X \to X$ be a non-decreasing mapping such that (1.1) is satisfied for all $x, y \in X$ with $x \preceq y$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Theorem 1.5 (see [9]) In addition to the hypothesis of Theorem 1.3 or Theorem 1.4, suppose that for every $x, y \in X$, there exist $u \in X$ such that $u \preceq x$ and $u \preceq y$. Then T has a unique fixed point.

In this paper, we establish some fixed point theorems for monotonic mapping involving rational expression in the framework of S-metric spaces endowed with a partial order using a class of pairs of functions satisfying certain assumptions.

Sedghi et al. [10] introduced a new generalized metric space called an S-metric space.

Definition 1.6 (see [10]) Let X be a non-empty set. An S-metric on X is a function $S: X^3 \to [0, +\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(S1). $S(x, y, z) \geq 0$;

(S2). $S(x, y, z) = 0$ if and only if $x = y = z$;

(S3). $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then S is called an S-metric on X and (X, S) is called an S-metric space.

The following is the intuitive geometric example for S-metric spaces.

Example 1.7 (see [10], Example 2.4) Let $X = \mathbb{R}^2$ and d be the ordinary metric on X. Put $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y \in \mathbb{R}^2$, that is, S is the perimeter of the triangle given by x, y, z. Then S is an S-metric on X.

Lemma 1.8 (see [10], Lemma 2.5) Let (X, S) be an S-metric space. Then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.

Lemma 1.9 (see [11], Lemma 1.6) Let (X, S) be an S-metric space. Then $S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$ and $S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$ for all $x, y, z \in X$.

Definition 1.10 (see [10]) Let X be an S-metric space.

(i) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ converges to x if and only if $\lim_{n\to\infty} S(x_n, x_n, x) = 0$. That is for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0, S(x_n, x_n, x) < \epsilon$ and we denote this by $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is called a Cauchy if $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$. That is, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $, m \geq n_0$, $S(x_n, x_n, x_m) < \epsilon$.

(iii) X is called complete if every Cauchy sequence in X is a convergent sequence.

From (see [10], Examples in page 260), we have the following.

Example 1.11

(a). Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$, is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$. Furthermore, the usual S-metric space $\mathbb{R}$ is complete.

(b). Let Y be a non-empty set of $\mathbb{R}$. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in Y$, is an S-metric on Y. If Y is a closed subset of the usual metric space $\mathbb{R}$, then the S-metric space Y is complete.

Lemma 1.12 (see [10], Lemma 2.11) Let (X, S) be an S-metric space. If the sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to x, then x is unique.

Lemma 1.13 (see [10], Lemma 2.12) Let (X, S) be an S-metric space. If $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} y_n = y$, then $\lim_{n\to+\infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Remark 1.14 (see [11]) It is easy to see that every D^*-metric (see [4]) is S-metric, but in general the converse is not true, see the following example.

Example 1.15 (see [11]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is S-metric on X, but it is not D^*-metric because it is not symmetric.

The following lemma shows that every metric space is an S-metric space.

Lemma 1.16 (see [11], Lemma 1.10) Let (X, d) be a metric space. Then we have

1. $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an S-metric on X.
2. $\lim_{n\to+\infty} x_n = x$ in (X, d) if and only if $\lim_{n\to+\infty} x_n = x$ in (X, S_d).
3. $\{x_n\}_{n=1}^{\infty}$ is Cauchy in (X, d) if and only if $\{x_n\}_{n=1}^{\infty}$ is Cauchy in (X, S_d).
4. (X, d) is complete if and only if (X, S_d) is complete.

In 2012, Sedghi et al. [10] asserted that an S-metric is a generalization of a G-metric, that is, every G-metric is an S-metric, see [10, Remarks 1.3] and [10, Remarks 2.2]. The Example 2.1 and Example 2.2 of Dung et al. [12] shows that this assertion is not correct. Moreover, the class of all S-metrics and the class of all G-metrics are distinct. For more results on S-metric spaces, see [11-12].

In this paper, we consider the following class of pairs of functions $\mathfrak{F}$.

Definition 1.17 (see [13]) A pair of functions (φ, ϕ) is said to belong to the class $\mathfrak{F}$, if they satisfy the following conditions:

(b1). $\varphi, \phi: [0, \infty) \to [0, \infty)$;

(b2). for $t, s \in [0, \infty)$, $\varphi(t) \leq \phi(s)$ then $t \leq s$;

(b3). for $\{t_n\}$ and $\{s_n\}$ sequence in $[0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = a$, if $\varphi(t_n) \le \phi(s_n)$ for any $n \in \mathbb{N}$, then $a = 0$.

Remark 1.18 (see [13], Remark 4) Note that, if $(\varphi, \phi) \in \mathfrak{F}$ and $\varphi(t) \le \phi(t)$, then $t = 0$, since we can take $t_n = s_n = t$ for any $n \in \mathbb{N}$ and by (b3) we deduce that $t = 0$.

Example 1.19 (see [13], Example 5) Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous and increasing function such that $\varphi(t) = 0$ if and only if $t = 0$ (these functions are known in the literature as altering distance functions). Let $\phi : [0,\infty) \to [0,\infty)$ be a non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and suppose that $\phi \le \varphi$. Then the pair $(\varphi, \varphi - \phi) \in \mathfrak{F}$.
An interesting particular case is when φ is the identity mapping, $\varphi = 1_{[0,\infty)}$ and $\phi: [0,\infty) \to [0,\infty)$ is a non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) \le t$ for any $t \in [0,\infty)$.

Example 1.20 (see [13], Example 6) Let S be the class of functions defined by

$$S = \{\alpha : [0,\infty) \to [0,1) : \{\alpha(t_n) \to 1 \Rightarrow t_n \to 0\}\}.$$

Let us consider the pairs of functions $(1_{[0,\infty)}, \alpha 1_{[0,\infty)})$, where $\alpha \in S$ and $\alpha 1_{[0,\infty)}$ is defined by $\left(\alpha 1_{[0,\infty)}\right)(t) = \alpha(t)t$, for $t \in [0,\infty)$. Then $(1_{[0,\infty)}, \alpha 1_{[0,\infty)}) \in \mathfrak{F}$.

Remark 1.21 (see [13], Remark 7) Suppose that $g : [0,\infty) \to [0,\infty)$ is an increasing function and $(\varphi, \phi) \in \mathfrak{F}$. Then it is easily seen that the pair $(g \circ \varphi, g \circ \phi) \in \mathfrak{F}$.

For more fixed point results with alternating distance function, see [14-19].

Definition 1.22 (see [20]) Let $(X, \preceq)$ be a partially ordered set and let $f, g: X \to X$ be two maps. Map$f$ is called g-non-decreasing if $gx \preceq gy$ implies $fx \preceq fy$ for all $x, y \in X$.

Definition 1.23 (see [21]) Let X be a non-empty set and let $f, g: X \to X$ be two maps. f and g are called to commute at $x \in X$ if $f(gx) = g(fx)$.

2. Main Results:

In this section, we investigate the fixed point problem on S-metric spaces. The following result states the existence of a fixed point of a map f on partially ordered S-metric spaces.

Theorem 2.1 Let $(X, \preceq)$ is a partially ordered set. Suppose that there exists an S-metric S on X such that (X, S) be a complete S-metric space. Let $f: X \to X$ be a non-decreasing map such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying

$$\varphi\big(S(fx, fx, fy)\big) \le max\left\{\phi\big(S(x, x, y)\big), \phi\left(\frac{S(y,y,fy)[1+S(x,x,fx)]}{1+S(fx,fx,fy)}\right)\right\}, \quad (2.1)$$

for all $x, y \in X$ with $x \preceq y$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof Choose $x_1 \in X$ such that $x_1 = fx_0$. Again, we can choose $x_2 \in X$ such that $x_2 = fx_1$. Continuing this process, we can choose a sequence $\{x_n\}$ in X such that

$$x_{n+1} = fx_n, \forall\, n \in \mathbb{N}. \qquad (2.2)$$

Since $x_0 \preceq fx_0$ and $x_1 = fx_0$, we have $x_0 \preceq x_1$. Since f is non-decreasing, we get $fx_0 \preceq fx_1$. By using (2.2), we have $x_1 \preceq x_2$. Again, since f is non-decreasing, we get $fx_1 \preceq fx_2$, that is, $x_2 \preceq x_3$. Continuing this process, we obtain

$$fx_n \preceq fx_{n+1}, x_{n+1} \preceq x_{n+2}, \forall\, n \in \mathbb{N}$$

Denote $\delta_n = S(fx_n, fx_n, fx_{n+1}), \forall\, n \in \mathbb{N}$. To prove that f has a fixed point. We consider two following cases.

Case 1. There exists n_0 such that $\delta_{n_0} = 0$. It implies that $x_{n_0} = fx_{n_0+1}$. By (2.2), we get $fx_{n_0+1} = x_{n_0+1}$. Therefore, x_{n_0+1} is a fixed point of f.

Case 2. Let $\delta_n > 0$ for all $n \in \mathbb{N}$. We will show that $\lim_{n\to\infty} \delta_n = 0$. Since $fx_{n-1} \prec fx_n$ for all $n \geq 1$, applying the contractive condition (2.1), we have

$$\begin{aligned}\varphi(\delta_n) &= \varphi(S(fx_n, fx_n, fx_{n+1}))\\ &\leq \max\left\{\phi\big(S(x_n, x_n, x_{n+1})\big), \phi\left(\frac{S(x_{n+1},x_{n+1},fx_{n+1})[1+S(x_n,x_n,fx_n)]}{1+S(fx_n,fx_n,fx_{n+1})}\right)\right\}\\ &= \max\left\{\phi\big(S(fx_{n-1}, fx_{n-1}, fx_n)\big), \phi\left(\frac{S(fx_n,fx_n,fx_{n+1})[1+S(fx_{n-1},fx_{n-1},fx_n)]}{1+S(fx_n,fx_n,fx_{n+1})}\right)\right\}\\ &= \max\left\{\phi(\delta_{n-1}), \phi\left(\frac{\delta_n[1+\delta_{n-1}]}{1+\delta_n}\right)\right\} \qquad (2.3)\end{aligned}$$

Now, we consider two following subcases.

Subcase 1. Consider

$$\max\left\{\phi(\delta_{n-1}), \phi\left(\frac{\delta_n[1+\delta_{n-1}]}{1+\delta_n}\right)\right\} = \phi(\delta_{n-1}) \qquad (2.4)$$

In this case from (2.3), we have

$$\varphi(\delta_n) \leq \phi(\delta_{n-1}) \qquad (2.5)$$

Since $(\varphi, \phi) \in \mathfrak{F}$, we deduce that $\delta_n \leq \delta_{n-1}$.

Subcase 2. If

$$\max\left\{\phi(\delta_{n-1}), \phi\left(\frac{\delta_n[1+\delta_{n-1}]}{1+\delta_n}\right)\right\} = \phi\left(\frac{\delta_n[1+\delta_{n-1}]}{1+\delta_n}\right) \qquad (2.6)$$

In this case from (2.3), we have

$$\varphi(\delta_n) \leq \phi\left(\frac{\delta_n[1+\delta_{n-1}]}{1+\delta_n}\right) \qquad (2.7)$$

Since $(\varphi, \phi) \in \mathfrak{F}$ and $\delta_n > 0$, we deduce that $\delta_n \leq \delta_{n-1}$.

The conclusions of two above subcases,

$$\delta_n \leq \delta_{n-1} \qquad (2.8)$$

It follows from (2.8) that the sequence $\{\delta_n\}$ of real numbers is monotone decreasing.Then there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = r. \qquad (2.9)$$

Now, we shall show that $r = 0$.

Denote $A = \{n \in \mathbb{N} :\ n \text{ satisfies } (2.4)\}$ and $B = \{n \in \mathbb{N} :\ n \text{ satisfies } (2.6)\}$.

From (2.3), we have $Card\ A = \infty$ or $Card\ B = \infty$. Let us suppose that $Card\ A = \infty$. Then from (2.3), we can find infinitely natural numbers n satisfying inequality (2.5) and since $(\varphi, \phi) \in \mathfrak{F}$, we infer from (2.9) and condition (b3) that $r = 0$. On the other hand, if $Card\ B = \infty$,, then from (2.3), we can find infinitely many $n \in \mathbb{N}$ satisfying inequality (2.7). Since $(\varphi, \phi) \in \mathfrak{F}$, we obtain

$$\delta_n \leq \frac{\delta_n[1+\delta_{n-1}]}{1+\delta_n}$$

for infinitely many $n \in \mathbb{N}$. Letting the limit as $n \to \infty$ and taking into account that (2.9), we deduce that $r \leq r\,(1+r)/(1+r)$ and consequently, we obtain $r = 0$.

Therefore

$$\lim_{n\to\infty} \delta_n = r = 0. \tag{2.10}$$

Now, we will show that $\{fx_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{fx_n\}$ is not a Cauchy sequence. Then given$\epsilon > 0$, we will construct a pair of subsequences $\{fx_{m_i}\}$ and $\{fx_{n_i}\}$ violating the following condition for least integer m_i such that $m_i > n_i > i$, where $i \in \mathbb{N}$:

$$\gamma_i = S\left(fx_{n_i}, fx_{n_i}, fx_{m_i}\right) \geq \epsilon \tag{2.11}$$

In addition, upon choosing the smallest possible m_i, we may assume that

$$S\left(fx_{n_i}, fx_{n_i}, fx_{m_i-1}\right) < \epsilon \tag{2.12}$$

From Lemma 1.1, Lemma 1.2, (2.11) and (2.12), we have

$$\begin{aligned}
\epsilon &\leq \gamma_i \\
&= S\left(fx_{n_i}, fx_{n_i}, fx_{m_i}\right) \\
&= S\left(fx_{m_i}, fx_{m_i}, fx_{n_i}\right) \\
&\leq 2S\left(fx_{m_i}, fx_{m_i}, fx_{m_i-1}\right) + S\left(fx_{n_i}, fx_{n_i}, fx_{m_i-1}\right) \\
&\leq 2S\left(fx_{m_i-1}, fx_{m_i-1}, fx_{m_i}\right) + S\left(fx_{n_i}, fx_{n_i}, fx_{m_i-1}\right) \\
&\leq \epsilon + 2\delta_{m_i-1}
\end{aligned} \tag{2.13}$$

On letting the limit as $i \to \infty$ in the above inequality, we obtain

$$\lim_{i\to\infty} \gamma_i = \epsilon \tag{2.14}$$

If we denote $\beta_i = S\left(fx_{n_i+1}, fx_{n_i+1}, fx_{m_i+1}\right)$, we notice that

$$\begin{aligned}
|\beta_i - \gamma_i| &= \left|S\left(fx_{n_i+1}, fx_{n_i+1}, fx_{m_i+1}\right) - \gamma_i\right| \\
&\leq 2S\left(fx_{n_i+1}, fx_{n_i+1}, fx_{n_i}\right) + S\left(fx_{m_i+1}, fx_{m_i+1}, fx_{n_i}\right) - \gamma_i \\
&= 2S\left(fx_{n_i}, fx_{n_i}, fx_{n_i+1}\right) + 2S\left(fx_{m_i+1}, fx_{m_i+1}, fx_{m_i}\right) - \gamma_i \\
&\leq 2\delta_{n_i} + 2S\left(fx_{m_i+1}, fx_{m_i+1}, fx_{m_i}\right) + S\left(fx_{n_i}, fx_{n_i}, fx_{m_i}\right) - \gamma_i \\
&= 2\delta_{n_i} + 2S\left(fx_{m_i}, fx_{m_i}, fx_{m_i+1}\right) + \gamma_i - \gamma_i \\
&= 2\delta_{n_i} + 2\delta_{m_i}
\end{aligned} \tag{2.15}$$

On making $i \to \infty$, we immediately obtain that:

$$\lim_{i\to\infty} \beta_i = \epsilon \tag{2.16}$$

It follows from (2.2) and (2.3) that $gx_{n_i+1} = fx_{n_i} \preceq fx_{m_i} = x_{m_i+1}$.Now using contractive condition (2.1), we get

$$\varphi(\beta_i) = \varphi\left(S\left(fx_{n_i+1}, fx_{n_i+1}, fx_{m_i+1}\right)\right)$$
$$\leq$$
$$max\left\{\phi\left(S\left(x_{n_i+1}, x_{n_i+1}, x_{m_i+1}\right)\right), \phi\left(\frac{S\left(x_{m_i+1}, x_{m_i+1}, fx_{m_i+1}\right)\left[1+S\left(x_{n_i+1}, x_{n_i+1}, fx_{n_i+1}\right)\right]}{1+S\left(fx_{n_i+1}, fx_{n_i+1}, fx_{m_i+1}\right)}\right)\right\}$$
$$=$$
$$max\left\{\phi\left(S\left(fx_{n_i}, fx_{n_i}, fx_{m_i}\right)\right), \phi\left(\frac{S\left(fx_{m_i}, fx_{m_i}, fx_{m_i+1}\right)\left[1+S\left(fx_{n_i}, fx_{n_i}, fx_{n_i+1}\right)\right]}{1+S\left(fx_{n_i+1}, fx_{n_i+1}, fx_{m_i+1}\right)}\right)\right\}$$
$$= max\left\{\phi(\gamma_i), \phi\left(\frac{\delta_{m_i}\left[1+\delta_{n_i}\right]}{1+\beta_i}\right)\right\} \tag{2.17}$$

Let us put

$$B = \{i \in \mathbb{N} : \varphi(\beta_i) \leq \phi(\gamma_i)\},$$
$$C = \left\{i \in \mathbb{N} : \varphi(\beta_i) \leq \phi\left(\frac{\delta_{m_i}\left[1+\delta_{n_i}\right]}{1+\beta_i}\right)\right\}.$$

By (2.17), we have $Card\ B = \infty$ or $Card\ C = \infty$. Let us suppose that $Card\ B = \infty$. Then there exists infinitely many $i \in \mathbb{N}$ satisfying inequality $\varphi(\beta_i) \leq \phi(\gamma_i)$ and since $(\varphi, \phi) \in \mathfrak{F}$, we have by letting the limit as $i \to \infty$, $\lim_{i\to\infty} \beta_i \leq \lim_{i\to\infty} \gamma_i$. We infer from (2.14) and (2.16) that $\epsilon = 0$. This is a contradiction. On the other hand, if $Card\ C = \infty$, then we can find infinitely many $i \in \mathbb{N}$ satisfying inequality $\varphi(\beta_i) \leq \phi\left(\frac{\delta_{m_i}\left[1+\delta_{n_i}\right]}{1+\beta_i}\right)$ and since $(\varphi, \phi) \in \mathfrak{F}$, we obtain $\beta_i \leq \frac{\delta_{m_i}\left[1+\delta_{n_i}\right]}{1+\beta_i}$. Om letting the limit as $i \to \infty$ and using (2.10) and (2.16) we get $\epsilon \leq 0$, which is a contradiction. Therefore, since in both possibilities $Card\ B = \infty$ and $Card\ C = \infty$, we obtain a contradiction, we deduce that $\{fx_n\}$ is a Cauchy sequence. From (2.1), we have $\{x_{n+1}\}$ is also a Cauchy sequence. Since X is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_n = u. \tag{2.18}$$

Now we will show that u is a fixed point of f. Since $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$ for all $n \in \mathbb{N}$. Applying contractive condition (2.1), we obtain for any $n \in \mathbb{N}$,

$$\varphi\left(S(fu, fu, fx_n)\right) \leq max\left\{\phi\left(S(u, u, x_n)\right), \phi\left(\frac{S(x_n, x_n, fx_n)[1+S(u,u,fu)]}{1+S(fu,fu,fx_n)}\right)\right\} \tag{2.19}$$

Put

$$E = \left\{n \in \mathbb{N} : \varphi\left(S(fu, fu, fx_n)\right) \leq \phi\left(S(u, u, x_n)\right)\right\},$$
$$F = \left\{n \in \mathbb{N} : \varphi\left(S(fu, fu, fx_n)\right) \leq \phi\left(\frac{S(x_n, x_n, fx_n)[1+S(u,u,fu)]}{1+S(fu,fu,fx_n)}\right)\right\}.$$

By (2.19), we have $Card\ E = \infty$ or $Card\ F = \infty$. Let us suppose that $Card\ E = \infty$.

Then there exists infinitely many $n \in \mathbb{N}$ satisfying inequality $\varphi(S(fu, fu, fx_n)) \leq \phi(S(u, u, x_n))$ and since $(\varphi, \phi) \in \mathfrak{F}$, letting the limit as $n \to \infty$ and using (2.18), we obtain $\lim_{n\to\infty} S(fu, fu, fx_n) = 0$, and consequently, we obtain $\lim_{n\to\infty} fx_n = fu$. The uniqueness of the limit, since $\lim_{n\to\infty} fx_n = u$, we have $fu = u$.

On the other hand, if $Card\ F = \infty$, we can find infinitely many $n \in \mathbb{N}$ satisfying inequality

$$\varphi(S(fu, fu, fx_n)) \leq \phi\left(\frac{S(x_n, x_n, fx_n)[1 + S(u, u, fu)]}{1 + S(fu, fu, fx_n)}\right) \quad (2.20)$$

Now, passing to the limit in

$$S(x_n, x_n, fx_n) \leq S(x_n, x_n, u) + S(x_n, x_n, u) + S(fx_n, fx_n, u) \quad (2.21)$$

as $n \to \infty$, we obtain $\lim_{n\to\infty} S(x_n, x_n, fx_n) = 0$. Since $(\varphi, \phi) \in \mathfrak{F}$, letting the limit as $n \to \infty$ in (2.20) and taking into account that $\lim_{n\to\infty} S(x_n, x_n, fx_n) = 0$, we deduce that $\lim_{n\to\infty} S(fu, fu, fx_n) = 0$ and consequently, we obtain $\lim_{n\to\infty} fx_n = fu$. Thus, we have $fu = u$. Therefore, in both the cases, u is a fixed point of f. This result finishes the proof.

By Theorem 2.1, we obtain the following corollaries.

Corollary 2.2 Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f: X \to X$ be a non-decreasing map such that

$$S(fx, fx, fy) \leq \alpha S(x, x, y) + \beta \frac{S(y, y, fy)[1 + S(x, x, fx)]}{1 + S(fx, fx, fy)}, \quad (2.22)$$

for all $x, y \in X$ with $x \preceq y$, where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof: Since

$$\begin{aligned} S(fx, fx, fy) &\leq \alpha S(x, x, y) + \beta \frac{S(y, y, fy)[1 + S(x, x, fx)]}{1 + S(fx, fx, fy)}, \\ &\leq (\alpha + \beta) \max\left\{S(x, x, y), \frac{S(y, y, fy)[1 + S(x, x, fx)]}{1 + S(fx, fx, fy)}\right\} \\ &= \max\left\{(\alpha + \beta) S(x, x, y), (\alpha + \beta) \frac{S(y, y, fy)[1 + S(x, x, fx)]}{1 + S(fx, fx, fy)}\right\} \end{aligned}$$

for all comparable elements $x, y \in X$, where $\alpha + \beta < 1$. This condition is a particular case of the contractive condition appearing in Theorem 2.1 with the pair of functions $(\varphi, \phi) = (1_{[0,\infty)}, (\alpha + \beta)1_{[0,\infty)}) \in \mathfrak{F}$, given by $\varphi = 1_{[0,\infty)}$ and $\phi = (\alpha + \beta)1_{[0,\infty)}$, (see Example 1.20). Furthermore, we relaxed the requirement of the continuity of mapping to prove the results.

Corollary 2.3 Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f: X \to X$ be a non-decreasing map such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying

$$\varphi\big(S(fx,fx,fy)\big) \le \phi\big(S(x,x,y)\big), \tag{2.23}$$

for all $x, y \in X$ with $x \preceq y$, where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Corollary 2.4 Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f: X \to X$ be a non-decreasing map such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying

$$\varphi\big(S(fx,fx,fy)\big) \le \phi\left(\frac{S(y,y,fy)[1+S(x,x,fx)]}{1+S(fx,fx,fy)}\right), \tag{2.24}$$

for all $x, y \in X$ with $x \preceq y$, where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Taking into account Example 1.19, we have the following corollary.

Corollary 2.5 Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f: X \to X$ be a non-decreasing map such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying

$$\varphi\big(S(fx,fx,fy)\big) \le max\Big\{\varphi\big(S(x,x,y)\big) - \phi\big(S(x,x,y)\big), \varphi\left(\frac{S(y,y,fy)[1+S(x,x,fx)]}{1+S(fx,fx,fy)}\right) - \phi\left(\frac{S(y,y,fy)[1+S(x,x,fx)]}{1+S(fx,fx,fy)}\right)\Big\} \tag{2.25}$$

for all $x, y \in X$ with $x \preceq y$, where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Corollary 2.5 has the following consequences.

Corollary 2.6 Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f: X \to X$ be a non-decreasing map such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying

$$\varphi\big(S(fx,fx,fy)\big) \le \varphi\big(S(x,x,y)\big) - \phi\big(S(x,x,y)\big), \tag{2.26}$$

for all $x, y \in X$ with $x \preceq y$, where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Corollary 2.7 Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f: X \to X$ be a non-decreasing map such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying

$$\varphi\big(S(fx,fx,fy)\big) \le \varphi\left(\frac{S(y,y,fy)[1+S(x,x,fx)]}{1+S(fx,fx,fy)}\right) - \phi\left(\frac{S(y,y,fy)[1+S(x,x,fx)]}{1+S(fx,fx,fy)}\right), \tag{2.27}$$

for all $x, y \in X$ with $x \preceq y$, where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Taking into account Example 1.20, we have the following corollary.

Corollary 2.8 Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f: X \to X$ be a non-decreasing map such that there exists $\alpha \in S$ satisfying

$$S(fx, fx, fy) \le max\{\alpha(S(x, x, y))S(x, x, y), \alpha\left(\frac{S(y,y,fy)[1+S(x,x,fx)]}{1+S(fx,fx,fy)}\right)\left(\frac{S(y,y,fy)[1+S(x,x,fx)]}{1+S(fx,fx,fy)}\right)\} \quad (2.28)$$

for all $x, y \in X$ with $x \preceq y$, where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

A consequence of Corollary 2.8 is the following corollary.

Corollary 2.9 Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f: X \to X$ be a non-decreasing map such that there exists $\alpha \in S$ satisfying

$$S(fx, fx, fy) \le \alpha(S(x, x, y))S(x, x, y) \quad (2.29)$$

for all $x, y \in X$ with $x \preceq y$, where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Corollary 2.10 Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f: X \to X$ be a non-decreasing map such that there exists $\alpha \in S$ satisfying

$$S(fx, fx, fy) \le \alpha\left(\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right)\left(\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right) \quad (2.30)$$

for all $x, y \in X$ with $x \preceq y$, where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

In what follows, we prove a sufficient condition for the uniqueness of the fixed point in Corollary 2.11.

Theorem 2.11 Suppose that: (a) hypothesis of Theorem 2.1 hold, (b) for each $x, y \in X$, there exists $z \in X$ that is comparable to x and y. Then f has a unique fixed point.

Proof: As in the proof of Corollary 2.11, we see that f has a fixed point. Now we prove that the uniqueness of the fixe point of f. Let u and v be two fixed points of f.

We consider the following two cases:

Case.1 u is comparable to v. Then $f^n u$ is comparable to $f^n v$ for all $n \in \mathbb{N}$. For all $a \in X$, applying contractive condition (2.31), we have

$$\varphi(S(u, u, v)) = \varphi(S(f^n u, f^n u, f^n v))$$
$$\le max\left\{\phi(S(f^{n-1}u, f^{n-1}u, f^{n-1}v)), \phi\left(\frac{S(f^{n-1}v, f^{n-1}v, f^n v)[1+S(f^{n-1}u, f^{n-1}u, f^n u)]}{1+S(f^n u, f^n u, f^n v)}\right)\right\}$$
$$= max\left\{\phi(S(u, u, v)), \phi\left(\frac{S(v,v,v)[1+S(u,u,u)]}{1+S(u,u,v)}\right)\right\} \quad (2.32)$$

Consider

$$max\left\{\phi\big(S(u,u,v)\big),\phi\left(\frac{S(v,v,v)[1+S(u,u,u)]}{1+S(u,u,v)}\right)\right\}=\phi\big(S(u,u,v)\big)$$

Then from (2.33), we have $\varphi\big(S(u,u,v)\big)\le\phi\big(S(u,u,v)\big)$. Since $(\varphi,\phi)\in\mathfrak{F}$, it follows that $S(u,u,v)=0$ and so $u=\ v$. If

$$max\left\{\phi\big(S(u,u,v)\big),\phi\left(\frac{S(v,v,v)[1+S(u,u,u)]}{1+S(u,u,v)}\right)\right\}$$
$$=\phi\left(\frac{S(v,v,v)[1+S(u,u,u)]}{1+S(u,u,v)}\right)$$

Then from (2.33), we have

$$\varphi\big(S(u,u,v)\big)\le\phi\left(\frac{S(v,v,v)[1+S(u,u,u)]}{1+S(u,u,v)}\right).$$

Then since $(\varphi,\phi)\in\mathfrak{F}$, we have $S(u,u,v)\le 0$ and so $u=\ v$. Therefore, in both cases we proved that $u=\ v$.

Case.2 u is not comparable to v. Then there exists $z\in X$ that is comparable to u and v. Now, we can define the sequence $\{z_n\}$ in X as follows: $z_0=z,\ fz_n=z_{n+1}$, $\forall\ n\in\mathbb{N}$. Since f is non-decreasing we have,

$$z_0\le z_n\le z_{n+1} \text{ and } \lim_{n\to\infty}S(z_n,z_n,z_{n+1})=0. \qquad (2.33)$$

As $u\le z_n$, putting $x=u$ and $y=z_n$ in the contractive condition (2.31), we get

$$\begin{aligned}\varphi\big(S(u,u,z_{n+1})\big)&=\varphi\big(S(fu,fu,fz_n)\big)\\&\le max\left\{\phi\big(S(u,u,z_n)\big),\phi\left(\frac{S(z_n,z_n,z_{n+1})[1+S(u,u,fu)]}{1+S(fu,fu,fz_n)}\right)\right\}\\&=max\left\{\phi\big(S(u,u,z_n)\big),\phi\left(\frac{S(z_n,z_n,z_{n+1})}{1+S(u,u,z_{n+1})}\right)\right\} \qquad (2.34)\end{aligned}$$

Let us denote

$$G=\left\{n\in\mathbb{N}:\varphi\big(S(u,u,z_{n+1})\big)\le\phi\big(S(u,u,z_n)\big)\right\}$$
$$H=\left\{n\in\mathbb{N}:\varphi\big(S(u,u,z_{n+1})\big)\le\phi\left(\frac{S(z_n,z_n,z_{n+1})}{1+S(u,u,z_{n+1})}\right)\right\}$$

Now we remark following again.

(1). If $Card\ G=\infty$, then from (2.34), we can find infinitely natural numbers n satisfying inequality

$$\varphi\big(S(u,u,z_{n+1})\big)\le\phi\big(S(u,u,z_n)\big).$$

Since $(\varphi,\phi)\in\mathfrak{F}$, it follows that the sequence $\{S(u,u,z_{n+1})\}$ is non-increasing and it has a limit $l\ge 0$. Since

$$\lim_{n\to\infty}S(u,u,z_{n+1})=\lim_{n\to\infty}S(u,u,z_n)=l$$

and $(\varphi,\phi)\in\mathfrak{F}$,we obtain $l\ =\ 0$.

(2). If $Card\ H=\infty$, then from (2.34), we can find infinitely natural numbers n satisfying inequality

$$\varphi\big(S(u,u,z_{n+1})\big)\le\phi\left(\frac{S(z_n,z_n,z_{n+1})}{1+S(u,u,z_{n+1})}\right).$$

Then since $(\varphi, \phi) \in \mathfrak{F}$, we have

$$S(u, u, z_{n+1}) \leq \frac{S(z_n, z_n, z_{n+1})}{1 + S(u, u, z_{n+1})}$$

Since $\lim_{n\to\infty} S(z_n, z_n, z_{n+1}) = 0$ and $\lim_{n\to\infty} S(u, u, z_{n+1}) = l$, on making $n \to \infty$ we have $l = 0$.

Therefore, in both cases we proved that

$$\lim_{n\to\infty} S(u, u, z_{n+1}) = l = 0.$$

In the same way it can be deduced that

$$\lim_{n\to\infty} S(v, v, z_{n+1}) = 0.$$

Therefore passing to the limit in

$$S(u, u, v) \leq S(u, u, z_{n+1}) + S(u, u, z_{n+1}) + S(v, v, z_{n+1})$$

as $n \to \infty$, we obtain $u = v$. That is, the fixed point is unique.

3. Example:

We give an example to demonstrate the validity of the above result.

Example 3.1 Let $X = \{1, 2, 3\}$ and let S be defined as follows.

$$S(1, 1, 1) = S(2, 2, 2) = S(3,3,3) = 0,$$
$$S(1, 2, 3) = S(1, 3, 2) = S(2, 1, 3) = S(3, 1, 2) = 4,$$
$$S(2, 3, 1) = S(3, 2, 1) = S(1, 1, 2) = S(1, 1, 3) = S(2, 2, 1) = S(3, 3, 1) = 2,$$
$$S(2, 2, 3) = S(3, 3, 2) = 6,$$
$$S(2, 3, 2) = S(3, 2, 2) = S(3, 2, 3) = S(2, 3, 3) = 3,$$
$$S(1, 2, 1) = S(2, 1, 1) = S(1, 3, 1) = S(3, 1, 1) = S(2, 1, 2) = S(1, 2, 2) = S(3, 1, 3) = S(1, 3, 3) = 1.$$

We have $S(x, y, z) \geq 0$ for all $x, y, z \in X$ and $S(x, y, z) = 0$ if and only if $x = y = z$. By simple calculations, we see that the inequality

$$S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$$

holds for all $x, y, z, a \in X$. Then S is an S-metric on X with the usual.

Consider the function $f : X \to X$ given as $fx = 1, \forall\, x \in X$. Define the functions $\varphi, \phi: [0, \infty) \to [0, \infty)$ as follows: for all $t \in [0, \infty)$, $\varphi(t) = \ln\left(\frac{1}{12} + \frac{5t}{12}\right)$ and $\phi(t) = \ln\left(\frac{1}{12} + \frac{3t}{12}\right)$. Then all assumptions of Theorem 2.1 are satisfied. Then Theorem 2.1 is applicable to f on S.

4. Conclusions:

In this article, we established some fixed point theorems for g-monotone maps involving rational expression in the framework of S-metric spaces endowed with a partial order using a class of pairs of functions satisfying certain assumptions. The presented theorems extend, generalize and improve many existing results on metric spaces to S-metric spaces in the literature. Our results may be the motivation to other authors for extending and improving these results to be suitable tools for their applications.

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Competing Interests

The authors declare that they have no competing interests.

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TRAFFIC FLOW AND SIMULTANEOUS LINEAR EQUATIONS

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Abstract:

In this article concept of linear equations is explained in light of Matrix and its properties. This concept is widely used to establish results when system of linear equation is given or found. To understand it required definitions and explanations are given in the form of different cases with appropriate examples
Keywords:

Linear equations, matrix, rank of a matrix, consistent and in-consistent system.

1. Introduction:

Linear algebra is a branch of mathematics that deals with linear equations and their representations in the vector space using matrices. It is the study of linear functions and vectors. Linear equations, matrices, and vector spaces are the most important components of this subject. System of simultaneous linear equations are important for studying and solving a large proportion of the problems in many topics in applied mathematics[1]Interest in modelling traffic flow has been around since the appearance of traffic jams. Ideally, if we can correctly predict the behavior of vehicle flow given an initial set of data. It is of particular interest in regions having high traffic density which may be caused by:

- •High volumes of vehicles in peak time
- • Accidents,
- • Closure of one or more lanes of the road etc.

In this article we will be focusing on a concept of linear algebra naming system of linear equations to regulate the traffic flow.

2. History of the topic:

The collections of equations:

$$(S)\quad \begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

where a_{ij} and b_j $(1 \leq i \leq m$ and $1 \leq j \leq n)$ belongs to the field F and are called as scalars. $x_1, x_2,...,x_n$ are n variables taking values from Field F, is called a system of m linear equations in n unknowns over the field F.

The matrix $(A)_{mxn}$ is called the coefficient matrix of the system (S).

If we let,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

then the system (S) can be rewritten as Ax = b, where A is the coefficient matrix and x,b arecolumn matrix of order nx1 and mx1 respectively

3. Solution of system of equations:

The values of the variable or unknown $x_1, x_2 \ldots x_n$ which satisfy the linear equations simultaneously is called as the solution to given system of linear equation. A solution to the system (S) is an n-tuple

$$s = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \in \mathsf{F}^n$$

A set containing all the solution to the given system of is called as the solution set of that system.

Consistent and inconsistent system:

If the set of solutions is non-empty we say that the system is consistent otherwise inconsistent.

Homogeneous and nonhomogeneous system of equations:

A system Ax = b consisting of m linear equations and n unknowns or variables is said to be homogeneous if b = 0. Otherwise, the system is said to be nonhomogeneous. Eg:

$ex+\pi y=0$
$\pi x+ey=0$, is a homogeneous system whereas
$2x+y=3$
$X+5y=6$, is a nonhomogeneous system

There are different methods of finding out the solutions to a system of linear equation

1. Cramer's rule
2. Matrix method
3. Rank method (here)

4. Solution of a nonhomogeneous system of linear equations

we have 2 categories of a non-homogeneous linear equation

1. Number of equation and number of variables are same
2. Number of equation and number of variables are different

For a matrix $(A)_{mxn}$ the rank is equal to the total number of linearly independent rows or columns.

We here concentrate on the technique of finding out the solutions of system of equations involves the rank of the coefficient matrix $Ax = b$ and the rank of the matrix [A:b] where [A:b]is called the augmented matrix to the system $Ax = b$.

Theorem

For a system $Ax = b$ of linear equations we say that the system is consistent if and only if rank(A) = rank[A:b]

Theorem

let Ax=b be a system oh m linear equations in n number of variables then:

Case 1 when $m>n$

1. If Rank [A]= Rank [A:b]=n, the system is consistent and has a unique solution.
2. If Rank [A]= Rank [A:b]$<$n, the system is consistent and has infinitely many solutions.
3. Rank [A] is not equal to Rank [A:b] the system is inconsistent and has no solution

Case 2 when $m<n$

rank[A]= rank[A:b]=r, where $r<m$ or $r=m$ implies $r<n$ then from case obe we say the system has infinitely many number of solutions.

Solution of a homogeneous system of linear equations

Clearly for a homogeneous system of linear equations rank [A]= rank[A:b] hence this system is always consistent i.e., for a homogenous system of linear equations zero vector is always a solution hence the solution set is never empty which implies that homogeneous systems are always consistent.

Case1: if rank[A]=r$<$**n**, the system has infinitely many solution

Case2: if rank[A]= **n**, the system has unique solution (n= number of variables)

Two systems of linear equations are called equivalent if they have the same solution set. A matrix is called as reduced row echelon form if it satisfies the following three conditions

(a) Any row containing a nonzero entry precedes any row in which all the entries are zero (if any).

(b) The first nonzero entry in each row is the only nonzero entry in the corresponding column.

(c) The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.

What is a traffic flow?

- In mathematics, traffic flow is the study of interactions between vehicles, drivers, and infrastructure (including highways, signage, and traffic control devices) with the aim of understanding and developing an optimal transport network with efficient movement of traffic and minimal traffic congestion problems.

Impact of traffic congestion

- Time consumption
- Chaos
- Bottle neck situation
- Misery to people
- Air Pollution
- Wear and tear on vehicles
- Encourage Road rage

5. Mathematical model

Model Assumptions

The following assumptions were made in order to ensure the smooth flow of the traffic:

1. Flow in equals flow out at each intersection
2. There should be only one way traffic

i.e., the streets must all be one-way with the arrows indicating the direction of traffic flow.

UNDERSTANDING THE MODEL MATHEMATICALLY WITH THE HELP OF AN EXAMPLE

Fig 1(network of one way street)

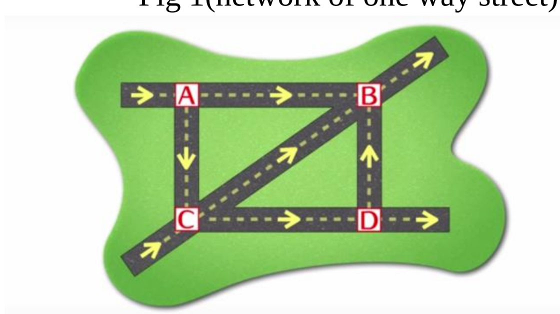


Fig 2(intersections)

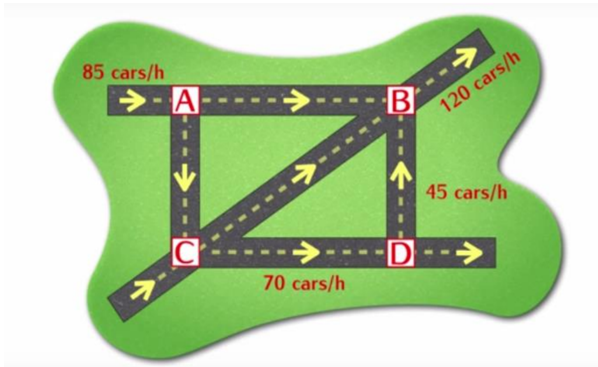


Fig 3(given initial data)

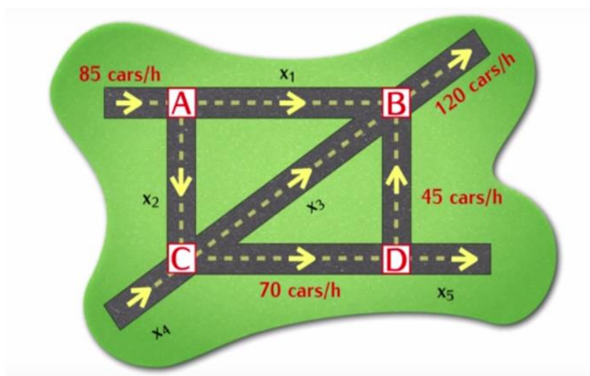


Fig 4(assignment of variable)

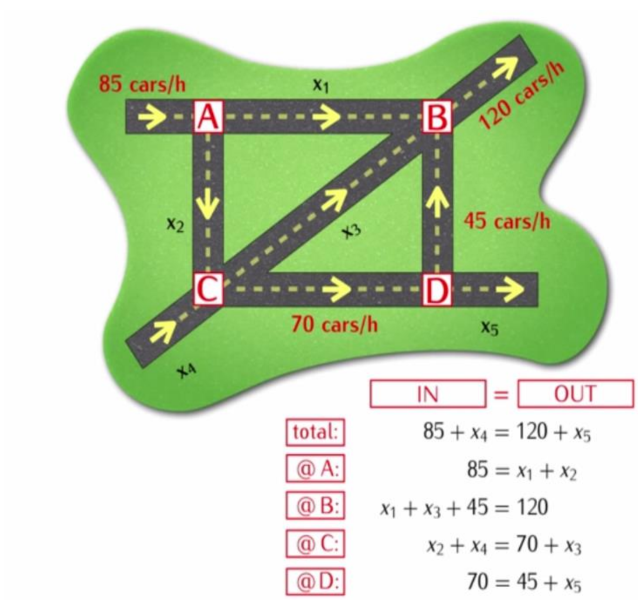
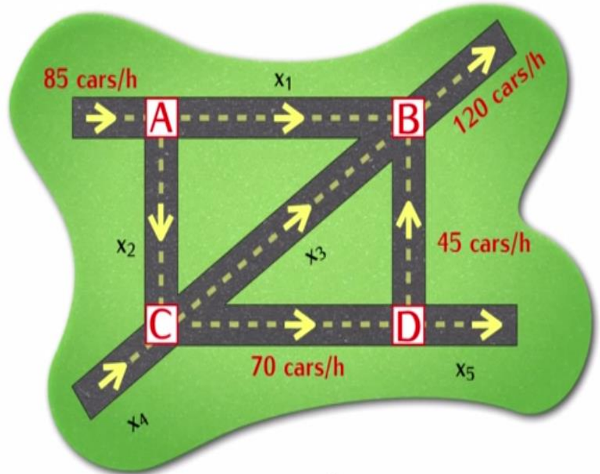


Fig 5(formulation of the problem)



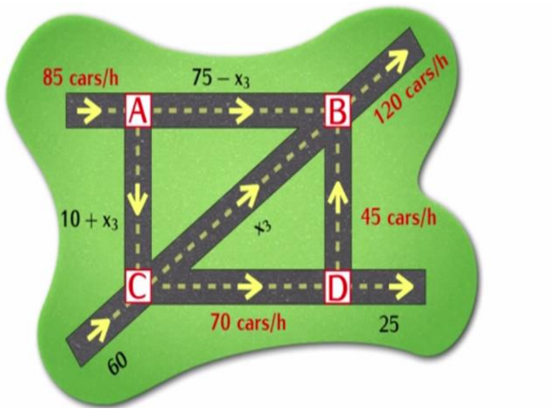
$$\begin{cases} x_4 - x_5 = 35 \\ x_1 + x_2 = 85 \\ x_1 + x_3 = 75 \\ x_2 - x_3 + x_4 = 70 \\ x_5 = 25 \end{cases}$$

Fig 6(formation of system of linear equation)

6. AUGMENTED MATRIX OF THE GIVEN SYSTEM OF EQUATIONS:

$$\begin{cases} x_4 - x_5 = 35 \\ x_1 + x_2 = 85 \\ x_1 + x_3 = 75 \\ x_2 - x_3 + x_4 = 70 \\ x_5 = 25 \end{cases} \qquad \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 1 & -1 & 35 \\ 1 & 1 & 0 & 0 & 0 & 85 \\ 1 & 0 & 1 & 0 & 0 & 75 \\ 0 & 1 & -1 & 1 & 0 & 70 \\ 0 & 0 & 0 & 0 & 1 & 25 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 1 & -1 & 35 \\ 1 & 1 & 0 & 0 & 0 & 85 \\ 1 & 0 & 1 & 0 & 0 & 75 \\ 0 & 1 & -1 & 1 & 0 & 70 \\ 0 & 0 & 0 & 0 & 1 & 25 \end{array}\right] \xrightarrow{\text{row reduction}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 75 \\ 0 & 1 & -1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & 60 \\ 0 & 0 & 0 & 0 & 1 & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$



$$\begin{cases} x_1 = 75 - x_3 \\ x_2 = 10 + x_3 \\ x_3 = free \\ x_4 = 60 \\ x_5 = 25 \end{cases}$$

Fig 7(solution)

7. CONCLUSION:

In the above article author has described how the concept of system of linear equations can be used to solve traffic flow problems. Above method can also be used to find unknown age, angles of trigonometry, calculation of speed, distance and time etc. with a given set of data.

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Encryption & Decryption of a message Involving Spiral Rotation Technique & Invertible Matrix

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ABSTRACT

The object of this paper is to introduce a new encryption algorithm involving Spiral rotation technique and invertible matrix. In the proposed algorithm firstly we apply the matrix multiplication under modulo 29, to get an intermediate cipher and then we apply the Spiral rotation technique that gives the final ciphertext. Using secret key matrix along with congruence modulo, the message can be encrypted and decrypted perfectly.

Keywords: Congruence, Spiral Rotation, Invertible Matrix, Encryption and Decryption.

1. INTRODUCTION:

In the presence of third parties, the sending of a message with secured coding is known as cryptographic technique [1]. Based on mathematical procedures and algorithms, the secured data transmission is done. There are multiple and multilevel encryption systems for ensuring the security. But when along with the complexity of the algorithm the security level increases the time for encryption and decryption, the sped and performances of these systems are also increases. In this research paper, we introduce a new encryption algorithm called 'Byte Spiral Rotation' along with invertible matrix that enhance the speed of the encryption scheme.

A square matrix AB said to bean invertible matrix iff there exists another square matrix B s.t. AB = BY = 1. It should be noted that all the square matrices are not invertible. If determinant value of a square matrix is non-zero, then the square matrix will be non-singular or invertible matrix.

2. LITERATURE REVIEW:

Bhati [2,3], Hamed [4] , Sani Isa [5], and Kumar [6] and other researchers introduced various algorithms for encryption and decryption of a message involving invertible matrix, Byte rotation and Spiral rotation techniques time to time. There is no any single and simple algorithms are sufficient for encryption used decryption of a message. Therefore, to obtain the better algorithm, researchers worked hard to remove the deficiency.

3. METHODOLOGY:

We use multiple encryption and multilevel encryption system for providing the sufficient security. Here we establish an algorithm model that having two steps. Firstly we apply the secret key matrix along with congruence modulo (Choose any prime number) and obtain an intermediate cipher. After that we apply the spiral rotation technique (agree both sender and receiver), to get final ciphertext.

For decryption the message we will use the reverse process of encryption along with spiral rotation technique and invertible matrix of congruence modulo p.

The numerical values for alphabets/character which are used in the paper are given in the following table:

Table –I

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14
O	P	Q	R	S	T	U	V	W	X	Y	Z	Space	
15	16	17	18	19	20	21	22	23	24	25	26	0	

4. ALGORITHM:

Encryption:

1. Consider a non-singular square matrix of order 4 as key matrix (sayK).
2. Arrange the character of plaintext in a block size of 16 bytes as 4×4 matrix. P (say).
3. Convert the alphabets (which are arranged in matrix form) into corresponding numeric values using Table I and assigned this resultant matrix by M (say).
4. Now calculate
   KP (mod p) = M (say),
5. Convert each entries of matrix M into corresponding alphabet/character by using Table-I, we get an intermediate ciphertext.
6. Obtain the transpose of M, Say M^T.
7. Apply the spiral rotation on the entries of M^T about the diagonal element by only one step from the first entry. For this technique sender and receiver should be agreed and maintain the secrecy also.
   Here we rotate the entries of

$$\begin{bmatrix} a_{11} & a_{12} a_{13} & a_{14} \\ a_{21} & a_{22} a_{23} & a_{24} \\ a_{31} & a_{32} a_{33} & a_{34} \\ a_{41} & a_{42} a_{43} & a_{44} \end{bmatrix}$$

as $a_{11} \to a_{21} \to a_{12} \to a_{13} \to a_{22} \to a_{31} \to a_{41} \to a_{32} \to a_{23} \to$
$a_{14} \to a_{24} \to a_{33} \to a_{42} \to a_{43} \to a_{34} \to a_{44} \to a_{11}$.

   We get the another matrix say M_{SR}
8. Convert all entries of M_{SR} into their corresponding alphabet/characters using Table I, to get final cipher text.
9. Send the cipher text, selected prime number as public key, Spiral rotation technique and integer n = 4 as private key to the receiver via secured channel.

Decryption Steps:

1. Consider the cipher text and arrange them in a square matrix of order 4 of block size of 16 bytes. After arranging convert them into their corresponding numeric values using Table-I, we get a resulting matrix D(say).
2. Rotating all the entries of matrix D about the diagonal elements by only one step form the last entry, we get after spiral rotation an another matrix say D_{SR}. For this technique sender and receiver should be agreed and maintain the secrecy also. Here we rotate the entries of D

$$D = \begin{bmatrix} a_{11} & a_{12} a_{13} & a_{14} \\ a_{21} & a_{22} a_{23} & a_{24} \\ a_{31} & a_{32} a_{33} & a_{34} \\ a_{41} & a_{42} a_{43} & a_{44} \end{bmatrix}$$

as follows: $a_{11} \rightarrow a_{44} \rightarrow a_{34} \rightarrow a_{43} \rightarrow a_{42} \rightarrow a_{33} \rightarrow a_{24} \rightarrow a_{14} \rightarrow a_{23} \rightarrow a_{41} \rightarrow a_{31} \rightarrow a_{22} \rightarrow a_{13} \rightarrow a_{12} \rightarrow a_{21} \rightarrow a_{11} \rightarrow a_{44}$.

3. Obtain the transpose of D_{SR}, say D_{SR}^T
4. Convert all the entries of D_{SR}^T into their corresponding alphabet/ characters using Table-I, we get intermediate plaintext (or cipher text).
5. Now calculate $K^{-1} D_{SR}^T \pmod p = p$(say),
   where p is a prime number.
6. Convert all the entries of p into their corresponding alphabet/ character using Table-I, we get another matrix of order 4×4 of block size 16 byte.
7. Arrange the alphabets (which are obtained in step 6) in row wise, we get the original plaintext.

Illustration:

Encryption Steps:

1. Consider a non-singular key matrix of order n (say n = 4) as key Matrix, say K given as follows:

$$K = \begin{bmatrix} 9 & 1 & 3 & 6 \\ 13 & 11 & 7 & 0 \\ 5 & 7 & 4 & 7 \\ 2 & 6 & 1 & 10 \end{bmatrix}, |K| \neq 0$$

and K^{-1} under modulo – p = 29 (say) is

$$K^{-1} = \begin{bmatrix} 20 & 2 & 19 & 24 \\ 21 & 21 & 14 & 24 \\ 21 & 9 & 9 & 13 \\ 19 & 18 & 13 & 26 \end{bmatrix}$$

where K and p are the public key used for encryption and K^{-1} and n are the private key used for decryption.

2. Let the plaintext be

"ASYMMETRICCIPHER"

3. Convert the plain text into block of size 4×4 matrix and their numeric equivalent using Table I, to get –

$$P = \begin{bmatrix} A & SY & M \\ M & ET & R \\ I & CC & I \\ P & HE & R \end{bmatrix} = \begin{bmatrix} 1 & 1925 & 13 \\ 13 & 5\ 20 & 18 \\ 9 & 3\ 3 & 9 \\ 16 & 8\ 5 & 18 \end{bmatrix}$$

4. Multiply the key matrix and plaintext matrix to generate, multiplication matrix under modulo 29, i.e.
   KP (mod 29) = M (say)
   =>M = KP (mod 29)

$$= \begin{bmatrix} 9 & 1\ 3 & 6 \\ 13 & 117 & 0 \\ 5 & 7\ 4 & 7 \\ 2 & 6\ 1 & 10 \end{bmatrix} \begin{bmatrix} 1 & 1925 & 13 \\ 13 & 5\ 20 & 18 \\ 9 & 3\ 3 & 9 \\ 16 & 8\ 5 & 18 \end{bmatrix} \text{(mod 29)}$$

$$= \begin{bmatrix} 0 & 1\ 23 & 9 \\ 16 & 4\ 15 & 24 \\ 12 & 2422 & 5 \\ 17 & 6\ 20 & 4 \end{bmatrix}$$

   Consider M as intermediate Ciphertext matrix, therefore intermediate Ciphertext by using Table I as follows:

   "AWIPDOXLXVEQFTD"

5. Find the transform of M i.e.

$$\Rightarrow M^T \text{ (say)} = \begin{bmatrix} 0 & 1612 & 17 \\ 1 & 4\ 24 & 6 \\ 23 & 1522 & 20 \\ 9 & 24\ 5 & 4 \end{bmatrix}$$

6. Rotating all the entries of MT about the diagonal elements by only one step from the first entry, we get by spiral rotation as follows:

$$M_{SR} \text{ (say)} = \begin{bmatrix} 4 & 1\ 16 & 24 \\ 0 & 1215 & 17 \\ 4 & 9\ 6 & 5 \\ 23 & 2224 & 20 \end{bmatrix}$$

7. Convert all the entries of M_{SR} into their corresponding alphabet/character using Table I, we get the following encrypted message as final ciphertext.

   "DAPX LOQDIFEWVXT"

Decryption Steps:

1. Consider the ciphertext

   "DAPX LOQDIFEWVXT"

2. Arrange it in block size of 16 bytes i.e. 4×4 matrix and convert them into their corresponding numeric values using Table I, we get –

$$D \text{ (say)} = \begin{bmatrix} 4 & 1\ 16 & 24 \\ 0 & 1215 & 17 \\ 4 & 9\ 6 & 5 \\ 23 & 2224 & 20 \end{bmatrix}$$

3. Rotating all the entries of matrix D about the diagonal elements, only one step from the last entry, we get after spiral rotation (as per agreement of sender and receiver) as follows:

$$D_{SR}\text{ (say)} = \begin{bmatrix} 0 & 1612 & 17 \\ 1 & 4\ 24 & 6 \\ 23 & 1522 & 20 \\ 9 & 24\ 5 & 4 \end{bmatrix}$$

4. Find the transpose of D_{SR} i.e.

$$D_{SR}^{T}\text{ (say)} = \begin{bmatrix} 0 & 1\ \ 23 & 9 \\ 16 & 4\ \ 15 & 24 \\ 12 & 2422 & 5 \\ 17 & 6\ 20 & 4 \end{bmatrix}$$

5. Convert all the entries of DSR into their corresponding alphabet/characters using Table I, we get the following intermediate plaintext (or ciphertext):

"AWIPDOXLXVEQFTD"

6. Now calculate

$K^{-1}D_{SR}^{T}$ (mod 29) = p (say)

$$\Rightarrow P = \begin{bmatrix} 20 & 2 & 19 & 24 \\ 21 & 21 & 14 & 24 \\ 21 & 9 & 9 & 13 \\ 19 & 18 & 13 & 26 \end{bmatrix} \begin{bmatrix} 0 & 1\ \ 23 & 9 \\ 16 & 4\ \ 15 & 24 \\ 12 & 2422 & 5 \\ 17 & 6\ 20 & 4 \end{bmatrix} = (\text{mod } 29)$$

$$= \begin{bmatrix} 1 & 1925 & 13 \\ 13 & 5\ 20 & 18 \\ 9 & 3\ 3 & 9 \\ 16 & 8\ 5 & 18 \end{bmatrix}$$

7. Convert the all entries of above matrix P into their corresponding alphabet/character using Table I, we get a matrix of order 4×4 of block size 16 byte as follows:

$$\begin{bmatrix} A & SY & M \\ M & ET & R \\ I & CC & I \\ P & HE & R \end{bmatrix}$$

8. Arrange the elements of above matrix in row wise, we get the following original plaint text as follows:

"ASYMMETRICCIPHER"

5. RESULT AND DISCUSSION:

In this paper we used invertible matrix congruent modulo 29 for encrypting and decrypting the message. Here, mathematical relations have ben logically implemented to keep the inform information secure from others (except receiver). Since here we use mathematical logic, therefore using of matrices is the strongest method among the other cryptographic technique.

It is very difficult to extract the original information due to chosen of spiral rotation technique along with invertible matrix congruent modulo of a prime number. Here, brute force attack is also difficult due to the key size of 16 bytes (or 128 bits).

6. CONCLUSION:

Since in this paper we use the spiral rotation technique and invertible matrix, therefore the proposed algorithm arise the strong security system and produced cipher text cannot be broken easily.

Here, we also generate the double encryption system firstly from invertible matrix and see secondly from spiral rotation technique. The information could be send and received safely by using above method without key matrix, congruence relation and spiral rotation technique the message could not be decrypt.

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On Fixed Points for Expansion Mappings in Quasi-Gauge Function Space

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ABSTRACT:

In this paper, some fixed point theorems for expansion mappings are proved in sequentially complete quasi-gauge function space generated by the family of pseudo metrics.

KEYWORDS:

Quasi-gauge function space, fixed Point, Expansion Mappings, etc.

MATHEMATICS SUBJECT CLASSIFICATION:

Primary 47H10, Secondary 54H25.

1. INTRODUCTION:

Quasi-gauge space was first developed by Reilly [8, 9]. It is one of the space in which Banach contraction principle has been carried over. A quasi-gauge structure for topological spaces (X,T) is a family P of pseudometrics on X such that T has a subbase, i.e., the family $\beta(X, P, \varepsilon)$ is the set $\{y \in X : p(x, y) < \varepsilon\}$. If the topological space (X,T) has a quasi –gauge structure P, it is called a quasi-gauge space and is denoted by (X, P).

2. PRELIMINARIES :

To eastablish our main result we need the following definitions:

Definition 2.1: Let X be a non-empty set and Y^x Reilly[8,9] be a quasi-gauge function space. A non-negative real valued function p defined on the function space $(X^x \times Y^x)$ having pointwise topology with the properties that :

(i) $p(f, g)(x) = 0$ if $f = g \in Y^x$ and

(ii) $p(f, g)(x) \leq p(f, h)(x) + p(h, g)(x)$ for all $f, g, h \in Y^x$

Is called a quasi-gauge metric.

Definition 2.2: A sequence $\{f_n\}$ in a quasi-gauge function space (Y^x, P) is called p-cauchy, if for every $p \in P$, there is an integer k, such that $p(f_m, f_n)(x) < \varepsilon$ for all $m, n \geq k$.

Definition 2.3: A quasi-gauge function space (Y^x, P) is called sequentially complete, if every p-cauchy sequence in Y^x converges in Y^x.

Definition 2.4: An operator T on a quasi-gauge function space (Y^x, P) into itself is said to be an expansion map, if $p(Tf, Tg)(x) \geq \lambda p(f, g)(x)$, for all $f, g \in Y^x$, $\lambda > 1$.
Throughout in this paper we use the symbole;

$$p(f, g)(x).\, p(f, g)(x) = p^2(f, g)(x)$$

3. MAIN RESULTS:
Theorem 3.1 : Let (Y^x, P) be a sequentially complete quasi-gauge function space gegerated by the family P of pseudo metricsand let T_1 and T_2 be any two operators on Y^x, such that

(3.1.1) T_1 and T_2 are commutes,

(3.1.2) $$\left[p\left(T_1^r(f), T_2^s(g)\right)(x)\right]^2 \geq \lambda \min\{[p(f, g)(x)]^2, \left[p\left(f, T_1^r(f)\right)(x)\right]^2, \left[p\left(g, T_2^s(g)\right)(x)\right]^2, \left[p\left(f, T_1^r(f)\right)(x)\right].[p(f,g)(x)], \left[p\left(g, T_2^s(g)\right)(x)\right].[p(f,g)(x)], \left[p\left(f, T_1^r(f)\right)(x)\right].\left[p\left(g, T_2^s(g)\right)(x)\right], \left[p\left(T_1^r(f), T_2^s(g)\right)(x)\right].[p(f,g)(x)]\}$$

Where r and s are positive integer and $\lambda > 1$. Then T_1 and T_2 have a fixed point in (Y^x, P).

PROOF: Define the sequence $\{f_n\}$ as follows**,**

$$f_0(x) = T_1^r(f_1)(x), \qquad f_{2n-2}(x) = T_1^r(f_{2n-1})(x)$$
$$f_1(x) = T_2^s(f_2)(x) \quad \text{and} \quad f_{2n-1}(x) = T_2^s(f_{2n})(x)$$

If, $f_m = f_{m-1}$ for some m, then f_m has a fixed point of T_1 and T_2 . Hence, without loss of generality we can assume that $f_n = f_{n-1}$ for every n. From (3.1.1), we have

$$[p(f_0, f_1)(x)]^2 = \left[p\left(T_1^r(f_1), T_2^s(f_2)\right)(x)\right]^2$$
$$\geq \lambda \min\{[p(f_1, f_2)(x)]^2, \left[p\left(f_1, T_1^r(f_1)\right)(x)\right]^2, \left[p\left(f_2, T_2^s(f_2)\right)(x)\right]^2, \left[p\left(f_1, T_1^r(f_1)\right)(x)\right].[p(f_1, f_2)(x)], \left[p\left(f_2, T_2^s(f_2)\right)(x)\right].[p(f_1, f_2)(x)], \left[p\left(f_1, T_1^r(f_1)\right)(x)\right].\left[p\left(f_2, T_2^s(f_2)\right)(x)\right], \left[p\left(T_1^r(f_1), T_2^s(f_2)\right)(x)\right].[p(f_1, f_2)(x)]\}$$
$$\geq \lambda \min\{[p(f_1, f_2)(x)]^2, [p(f_1, f_0)(x)]^2, [p(f_2, f_1)(x)]^2, [p(f_1, f_0)(x)].[p(f_1, f_2)(x)], [p(f_2, f_1)(x)].[p(f_1, f_2)(x)], [p(f_1, f_0)(x)].[p(f_2, f_1)(x)], [p(f_0, f_1)(x)].[p(f_1, f_2)(x)]\}$$

Thus,

(3.1.3) $$[p(f_0, f_1)(x)]^2 \geq \lambda \min\{[p(f_1, f_2)(x)]^2, [p(f_1, f_0)(x)].[p(f_1, f_2)(x)]\}$$

Then from (3.1.3) we have,

Case I : If $[p(f_1, f_2)(x)]^2$ is minimum, then

$$[p(f_0, f_1)(x)]^2 \geq \lambda [p(f_1, f_2)(x)]^2, \text{ i.e.}$$

(3.1.4) $$[p(f_1, f_2)(x)] \leq \frac{1}{\sqrt{\lambda}}[p(f_0, f_1)(x)], \text{ as } \lambda > 1.$$

Case II : If $[p(f_1, f_0)(x)].[p(f_1, f_2)(x)]$ is minimum, then

$$[p(f_0, f_1)(x)]^2 \geq \lambda\, [p(f_1, f_0)(x)].[p(f_1, f_2)(x)]$$

Or, $[\mathrm{p}(f_0, f_1)(x)] \geq \lambda\,[\mathrm{p}(f_1, f_2)(x)]$,

(3.1.5) $[\mathrm{p}(f_1, f_2)(x)] \leq \frac{1}{\lambda}[\mathrm{p}(f_0, f_1)(x)] \leq \frac{1}{\sqrt{\lambda}}[\mathrm{p}(f_0, f_1)(x)]$, as $\lambda > 1$.

Therefore from (3.1.3), (3.1./4) and (3.1.5), we have

$$[\mathrm{p}(f_1, f_2)(x)] \leq \frac{1}{\sqrt{\lambda}}[\mathrm{p}(f_0, f_1)(x)]$$

Hence in general,

$$[\mathrm{p}(f_{2n}, f_{2n+1})(x)] \leq \left(\frac{1}{\sqrt{\lambda}}\right)^{2n}[\mathrm{p}(f_0, f_1)(x)] \to 0 \text{ as } n \to \infty.$$

Hence, $\{f_n\}$ is a Cauchy sequence. Since Y^x is sequentially complete, there exists $u \in Y^x$, such that $\lim_{n\to\infty} f_n = u$, and so we have

$$\lim_{n\to\infty} T_1^r(f_{2n-1}) = u \quad \text{and} \quad \lim_{n\to\infty} T_2^s(f_{2n}) = u$$

Thus, u is a common fixed point of T_1 and T_2. This completes the proof.

Initially, Maia [5], have proved fixed point theorems in space having two different matrices. On the same line we shall obtain a result having two different quasi-gauge function space.

Theorem 3.2 : Let Y^x be a sequentially complete quasi-gauge function space with two quasi-gauge structurs P and P_1, such that

(3.2.1) $P_1(f, g)(x) = P(f, g)(x)$,

(3.2.2) T_1 and T_2 are continuous w. r. t. P_1,

(3.1.3) Y^x is sequentially complete w. r. t. P_1 and

(3.1.4) T_1 and T_2 satisfies conditions (3.1.1) and (3.1.2) w. r. t. P.

Then T_1 and T_2 have a fixed point.

PROOF: Define the sequence $\{f_n\}$ as follows**,**

$$f_0(x) = T_1^r(f_1)(x), \qquad f_{2n-2}(x) = T_1^r(f_{2n-1})(x)$$
$$f_1(x) = T_2^s(f_2)(x) \quad \text{and} \quad f_{2n-1}(x) = T_2^s(f_{2n})(x)$$

Then proceeding as in the proof of theorem 3.1 with similar arguments, we get

$$[\mathrm{P}(f_{2n}, f_{2n+1})(x)] \leq \left(\frac{1}{\sqrt{\lambda}}\right)^{2n}[\mathrm{P}_1(f_0, f_1)(x)]$$

Since, $[\mathrm{P}_1(f, g)(x)] \leq \mathrm{P}(f, g)(x)$, we have

$$[P_1(f_{2n}, f_{2n+1})(x)] \leq [\mathrm{P}(f_{2n}, f_{2n+1})(x)]$$
$$\leq \left(\frac{1}{\sqrt{\lambda}}\right)^{2n}[\mathrm{P}_1(f_0, f_1)(x)] \to 0 \text{ as } n \to \infty.$$

Hence, $\{f_n\}$ is a Cauchy sequence w.r.t. P_1. Since Y^x is sequentially complete w.r.t. P_1, there exists $u \in Y^x$, such that $\lim_{n\to\infty} f_n = u$. Also, since T_1 and T_2 are continuous w. r. t. P_1, we have

$$u = \lim_{n\to\infty} f_{2n+1} \quad \text{implies that,}$$
$$\lim_{n\to\infty} T_1(f_{2n+1}) = T_1 \lim_{n\to\infty}(f_{2n+1}) = T_1 u,$$

Similarly, $u = \lim_{n\to\infty} f_{2n}$ implies that,

$$\lim_{n\to\infty} T_2(f_{2n}) = T_2 \lim_{n\to\infty}(f_{2n}) = T_2 u,$$

Thus, u is a common fixed point of T_1 and T_2. This completes the proof.

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